Coherent systems of finite support iterations

Diana Carolina Montoya
Joint work with: Vera Fischer, Sy-David Friedman and Diego Mejía

Kurt Gödel Research Center. University of Vienna.
RIMS workshop on infinite combinatorics and forcing theory

December 1st, 2016

Contents

Contents

Contents

Section 1

Constellations in Cichoń's diagram

The problem

Given a constellation from cardinals in Cichoń's diagram, is it possible to find a model where, in addition $\mathfrak{a}$ *can be decided?*

The unbounding and dominating numbers, $\mathfrak{b}$ and $\mathfrak{d}$.

Definition

If f, g are functions in ω^ω, we say that $f <^* g$, if there exists an $n \in \omega$ such that for all $m > n$, $f(m) < g(m)$. In this case, we say that g eventually dominates f.

Definition

Let $\mathfrak{F}$ be a family of functions in ω^ω.

- *$\mathfrak{F}$ is dominating, if for all $g \in \omega^\omega$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.*
- *$\mathfrak{F}$ is unbounded, if for all $g \in \omega^\omega$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.*

Definition

- $\mathfrak{b} = \min\{|\mathfrak{F}| \colon \mathfrak{F}$ *is an unbounded family of functions in* $\omega^\omega\}$.
- $\mathfrak{d} = \min\{|\mathfrak{F}| \colon \mathfrak{F}$ *is a dominating family of functions in* $\omega^\omega\}$.

Cardinal Invariants Associated to an Ideal

Let $\mathcal{I}$ be a non-trivial σ-ideal on a set X:

Definition

- *The additivity number:*

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}.$$

- *The covering number:*

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathcal{J} = X\}.$$

Definition

- *The cofinality number:*

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{J}| \colon \mathcal{J} \subseteq \mathcal{I} \textit{ and for all } M \in \mathcal{I} \textit{ there is a } J \in \mathcal{J} \textit{ with } M \subseteq J\}.$$

- *The uniformity number:*

$$\mathrm{non}(\mathcal{I}) = \min\{|Y| \colon Y \subset X \textit{ and } Y \notin \mathcal{I}\}.$$

Cichoń's Diagram

$$
\begin{tikzcd}
 & \operatorname{cov}\mathcal{N} \arrow[r] & \operatorname{non}\mathcal{M} \arrow[r] & \operatorname{cof}\mathcal{M} \arrow[r] & \operatorname{cof}\mathcal{N} \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \operatorname{add}\mathcal{N} \arrow[r] \arrow[uu] & \operatorname{add}\mathcal{M} \arrow[r] \arrow[u] & \operatorname{cov}\mathcal{M} \arrow[r] \arrow[u] & \operatorname{non}\mathcal{N} \arrow[uu] &
\end{tikzcd}
$$

The method of matrix iterations I

1. It was introduced by Blass and Shelah, to show that consistently $\mathfrak{u} < \mathfrak{d}$, where $\mathfrak{u}$ is the ultrafilter number and $\mathfrak{d}$ is the dominating number.
2. Further developed fron Brendle-Fischer, who introduced the terminology *matrix iteration* for the first time. They used the method to show that if $\kappa < \lambda$ are arbitrary regular uncountable cardinals then there is a generic extension in which $\mathfrak{a} = \mathfrak{b} = \kappa < \mathfrak{s} = \lambda$. Here $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{s}$ denote the almost disjointness, bounding and splitting numbers respectively.
3. Later, classical preservation properties for matrix iterations were improved by Mejía to provide several examples of models where the cardinals in Cichoń's diagram assume many different values, in particular, a model with 6 different values.

An example

Theorem (Mejía)

Let V be a model of ZFC *and fix regular uncountable cardinals $\theta_0 \leq \theta_1 \leq \kappa \leq \mu$ and let $\lambda \geq \mu$ to be a cardinal. Then there is a cardinal preserving generic extension in which:* $\text{add}(\mathcal{N}) = \theta_0$, $\text{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \kappa$, $\mathfrak{d} = \mu$ *and* $\text{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Idea of the proof: Blackboard

To remember:
Full generics and restricted generics.

Section 2

Preservation of mad families

The almost disjointness number

Definition

Two sets A and $B \in \mathcal{P}(\omega)$ are called almost disjoint if $A \cap B$ is finite. We say that an infinite family of sets $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is almost disjoint if all its elements are pairwise almost disjoint. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called a maximal almost disjoint (abbreviated mad) if it is almost disjoint and is not properly included in another such family.

Definition

$\mathfrak{a} = \min\{|\mathfrak{A}| \colon \mathfrak{A}$ *is a mad family of subsets of* $\omega\}$.

Preservation of mad families

Definition (Hechler-type almost disjoint families)

For a set Ω define the poset
$\mathbb{H}_\Omega := \{p : F_p \times n_p \to 2 : F_p \in [\Omega]^{<\aleph_0}$ *and* $n_p < \omega\}$. *The order is given by* $q \leq p$ *if and only if* $p \subseteq q$ *and, for any* $i \in n_q \setminus n_p$, *there is* *__at most one__* $z \in F_p$ *such that* $q(z, i) = 1$.

If G is $\mathbb{H}_\Omega$-generic over V then $A = A_G := \{a_z : z \in \Omega\}$ is an a.d. family where $a_z \subseteq \omega$ is defined as $i \in a_z$ if and only if $p(z, i) = 1$ for some $p \in G$. Moreover, $V[G] = V[A]$ and, when Ω is uncountable, A is mad in $V[G]$.

Some background I

1. Kunen: Under CH there is a mad family that remains maximal upon adding a Cohen real.
2. Steprans: there is a mad family which remains maximal upon adding $\aleph_1$-many Cohen reals.
3. Zhang proved that, under CH there is a mad family in the ground model that stays mad after a finite support iteration of $\mathbb{E}$.
4. Brendle and Fischer: defined a property that is iterable and used to preserve Hechler style mad families added in the first column of a matrix iteration of ccc posets.

Definition (Brendle-Fischer)

Let $A = \langle a_z \rangle_{z\in\Omega} \in M$ be a family of infinite subsets of ω and $a^* \in [\omega]^\omega$ (not necessarily in M). Say that a^* diagonalizes M outside A if, for all $h \in M$, $h : \omega \times [\Omega]^{<\omega} \to \omega$ and for any $m < \omega$, there are $i \geq m$ and $F \in [\Omega]^{<\omega}$ such that $[i, h(i, F)) \setminus \bigcup_{z\in F} a_z \subseteq a^*$.

Theorem

Let κ be an uncountable regular cardinal. After forcing with $\mathbb{H}_\kappa$, any finite support iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi \rangle_{\xi<\pi}$ where each iterand is either

(i) *in $\{\mathbb{C}, \mathbb{E}\} \cup \mathcal{R}$ or*

(ii) *a ccc poset of size $< \kappa$*

preserves the mad family added by $\mathbb{H}_\kappa$.

Section 3

Coherent system of finite support iterations

- The method provides a general framework that generalizes the method of matrix iterations of ccc posets introduced by Blass and Shelah.
- It was motivated by the necessity of adding a third dimension, in order to control the cardinal invariant $\mathfrak{a}$.

Coherent system of finite support iterations I

A *coherent system (of finite support iterations)* **s** is composed by the following objects:

(I) a partial ordered set $I^{\mathbf{s}}$ and an ordinal $\pi^{\mathbf{s}}$,

(II) a system of posets $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi \leq \pi^{\mathbf{s}} \rangle$ such that

  (i) $\mathbb{P}^{\mathbf{s}}_{i,0} < \cdot \mathbb{P}^{\mathbf{s}}_{j,0}$ whenever $i \leq j$ in $I^{\mathbf{s}}$, and

  (ii) $\mathbb{P}^{\mathbf{s}}_{i,\eta}$ is the direct limit of $\langle \mathbb{P}^{\mathbf{s}}_{i,\xi} : \xi < \eta \rangle$ for each limit $\eta \leq \pi^{\mathbf{s}}$,

(III) a sequence $\langle \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} : i \in I^{\mathbf{s}}, \xi < \pi^{\mathbf{s}} \rangle$ where each $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ is a $\mathbb{P}^{\mathbf{s}}_{i,\xi}$-name for a poset, $\mathbb{P}^{\mathbf{s}}_{i,\xi+1} = \mathbb{P}^{\mathbf{s}}_{i,\xi} * \dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi}$ and $\mathbb{P}^{\mathbf{s}}_{j,\xi}$ forces $\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} < \cdot_{V^{\mathbb{P}^{\mathbf{s}}_{i,\xi}}} \dot{\mathbb{Q}}^{\mathbf{s}}_{j,\xi}$ whenever $i \leq j$ in $I^{\mathbf{s}}$ and $\mathbb{P}^{\mathbf{s}}_{i,\xi} < \cdot \mathbb{P}^{\mathbf{s}}_{j,\xi}$

We consider the following particular cases.

1. When $I^{\mathbf{s}}$ is a well-ordered set, we say that **s** is a *2D-coherent system (of finite support iterations)*.
2. If $I^{\mathbf{s}}$ is of the form $\{i_0, i_1\}$ ordered as $i_0 < i_1$, we say that **s** is a *coherent pair (of finite support iterations)*.
3. If $I^{\mathbf{s}} = \gamma^{\mathbf{s}} \times \delta^{\mathbf{s}}$ where $\gamma^{\mathbf{s}}$ and $\delta^{\mathbf{s}}$ are ordinals and the order of $I^{\mathbf{s}}$ is defined as $(\alpha, \beta) \leq (\alpha', \beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$, we say that **s** is a *3D-coherent system (of finite support iterations)*.

Standard coherent system of FS iterations I

A coherent system of FS iterations $\mathbf{s}$ is *standard* if

(I) it consists, additionally, of:

(i) a partition $\langle S^{\mathbf{s}}, C^{\mathbf{s}} \rangle$ of $\pi^{\mathbf{s}}$,

(ii) a function $\Delta^{\mathbf{s}} : C^{\mathbf{s}} \to I^{\mathbf{s}}$ so that $\Delta^{\mathbf{s}}(i)$ is not maximal in $I^{\mathbf{s}}$ for all $i \in C^{\mathbf{s}}$,

(iii) a sequence $\langle \mathbb{S}^{\mathbf{s}}_{\xi} : \xi \in S^{\mathbf{s}} \rangle$ where each $\mathbb{S}^{\mathbf{s}}_{\xi}$ is either a Suslin ccc poset or a random algebra, and

(iv) a sequence $\langle \dot{\mathbb{Q}}^{\mathbf{s}}_{\xi} : \xi \in C^{\mathbf{s}} \rangle$ such that each $\dot{\mathbb{Q}}^{\mathbf{s}}_{\xi}$ is a $\mathbb{P}^{\mathbf{s}}_{\Delta^{\mathbf{s}}(\xi),\xi}$-name of a poset which is forced to be ccc by $\mathbb{P}^{\mathbf{s}}_{i,\xi}$ for all $i \geq \Delta^{\mathbf{s}}(\xi)$ in $I^{\mathbf{s}}$, and

Standard coherent system of FS iterations II

(II) it is satisfied, for any $i \in I^{\mathbf{s}}$ and $\xi < \pi^{\mathbf{s}}$, that

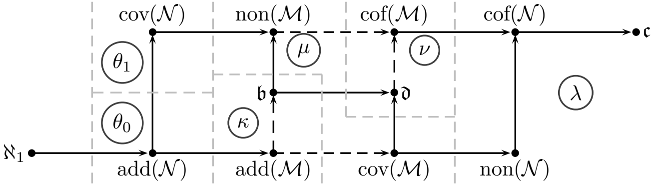
$$\dot{\mathbb{Q}}^{\mathbf{s}}_{i,\xi} = \begin{cases} (\mathbb{S}^{\mathbf{s}}_{\xi})^{V^{\mathbf{s}}_{i,\xi}} & \text{if } \xi \in S^{\mathbf{s}} \\ \dot{\mathbb{Q}}^{\mathbf{s}}_{\xi} & \text{if } \xi \in C^{\mathbf{s}} \text{ and } i \geq \Delta^{\mathbf{s}}(\xi), \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

The main theorem

Theorem

Assume $\lambda^{<\theta_1} = \lambda$. *Then, there is a ccc poset forcing* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

Cichoń's diagram as in Theorem 3.



More constellations...

Theorem

Assume $\lambda^{<\theta_0} = \lambda$. Then, for any of the statements below, there is a ccc poset forcing it.

(a) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \mu$ *for* $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, $\mathfrak{d} = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.

(b) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \kappa$, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$, $\mathrm{non}(\mathcal{N}) = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.

(c) $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \nu$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.

An application with well-orders of the reals

Theorem

In L, let $\theta_0 < \theta_1 < \kappa < \mu < \nu < \lambda$ *be uncountable regular cardinals and, in addition,* $\lambda < \aleph_\omega$*. Then there is a cardinals preserving forcing extension of the constructible universe, L, in which there is a* Δ^1_3*-well-order of the reals and in addition* $\mathrm{add}(\mathcal{N}) = \theta_0$, $\mathrm{cov}(\mathcal{N}) = \theta_1$, $\mathfrak{b} = \mathfrak{a} = \kappa$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mu$, $\mathfrak{d} = \nu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c} = \lambda$.

References I

[1] Jörg Brendle.
Larger cardinals in Cichoń's diagram.
J. Symbolic Logic, 56(3):795–810, 1991.

[2] Jörg Brendle and Vera Fischer.
Mad families, splitting families and large continuum.
J. Symbolic Logic, 76(1):198–208, 2011.

[3] Vera Fischer and Sy David Friedman.
Cardinal characteristics and projective wellorders.
Ann. Pure Appl. Logic, 161(7):916–922, 2010.

[4] Vera Fischer, Sy-David Friedman, Diego Mejía, and Diana Carolina Montoya.
Coherent systems of finite support iterations.
In preparation.

[5] Vera Fischer, Sy David Friedman, and Lyubomyr Zdomskyy.
Projective wellorders and mad families with large continuum.
Ann. Pure Appl. Logic, 162(11):853–862, 2011.

[6] Vera Fischer, Sy David Friedman, and Lyubomyr Zdomskyy.
Cardinal characteristics, projective wellorders and large continuum.
Ann. Pure Appl. Logic, 164(7-8):763–770, 2013.

[7] Martin Goldstern, Diego Alejandro Mejía, and Saharon Shelah.
The left side of Cichoń's diagram.
Proc. Amer. Math. Soc.
Accepted.

References II

[8] Stephen H. Hechler.
Short complete nested sequences in $\beta N \setminus N$ and small maximal almost-disjoint families.
General Topology and Appl., 2:139–149, 1972.

[9] Haim Judah and Saharon Shelah.
The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing).
J. Symbolic Logic, 55(3):909–927, 1990.

[10] Diego Alejandro Mejía.
Matrix iterations and Cichon's diagram.
Arch. Math. Logic, 52(3-4):261–278, 2013.

[11] Diego Alejandro Mejía.
Models of some cardinal invariants with large continuum.
Kyōto Daigaku Sūrikaiseki Kenkyūsho Kōkyūroku, (1851):36–48, 2013.

[12] Arnold W. Miller.
Some properties of measure and category.
Trans. Amer. Math. Soc., 266(1):93–114, 1981.