Boolean valued second order logic

Daisuke Ikegami

Tokyo Denki University

Joint work with Jouko Väänänen

1st of December, 2016

In this talk...

We will work on 2nd-order logic.

Review; 1st-order logic

1st-order logic enjoys several nice properties:

- Completeness Theorem
- The set of (Gödel numbers of) valid sentences is Σ^0_1.
- Compactness Theorem
- Löwenheim-Skolem-Tarski Theorem

Review; 1st-order logic

1st-order logic enjoys several nice properties:

- Completeness Theorem
- The set of (Gödel numbers of) valid sentences is Σ^0_1.
- Compactness Theorem
- Löwenheim-Skolem-Tarski Theorem

How about 2nd-order logic?

2nd-order logic; Two semantics

1. Henkin semantics: Very simple (very week), essentially the same as 1st-order logic
2. Full semantics: Highly complex (very powerful), does NOT enjoy completeness, compactness.

Full semantics

Full semantics = semantics with full 2nd-order structures

Definition

Full 2nd-order structures are those of the form $M = (A, \mathcal{P}(A), \ldots)$.

Full semantics

Full semantics $=$ semantics with full 2nd-order structures

Definition

Full 2nd-order structures are those of the form $M = (A, \mathcal{P}(A), \ldots)$.

Theorem (Väänänen?)

The set of valid 2nd-order sentences with full semantics is Π_2-complete in the language of set theory.

Full semantics

Full semantics $=$ semantics with full 2nd-order structures

Definition

Full 2nd-order structures are those of the form
$M = (A, \mathcal{P}(A), \ldots)$.

Theorem (Väänänen?)

The set of valid 2nd-order sentences with full semantics is Π_2-complete in the language of set theory.

Point: There is a 2nd-order sentence ϕ_0 in the language $\mathcal{L} = \{R\}$ such that for any 2nd-order full $\mathcal{L}$-structure $M = (A, \mathcal{P}(A), R^M)$,

$$M \models \phi_0 \iff \text{for some } \lambda \text{ with } \lambda = |V_\lambda|,\ (A, R^M) \simeq (V_\lambda, \in).$$

Full semantics ctd.

Theorem (Folklore?)

There is a 2nd-order sentence ϕ_1 in the language $\mathcal{L}$ of Peano Arithmetic (PA) such that for any 2nd-order full $\mathcal{L}$-structure $M = (A, \mathcal{P}(A), \ldots)$,

$$M \vDash \phi_1 \iff (A, \ldots) \simeq (\mathbb{N}, +, \times, 1).$$

In particular, Compactness Theorem fails for 2nd-order logic with full semantics.

In this talk...

We introduce Boolean valued semantics for 2nd-order logic and compare it with full semantics w.r.t. the following 3 points:

1. Complexity of the validity
2. How much Compactness Theorem holds (or fails)
3. The construction of Gödel's L w.r.t. the corresponding definability

Boolean valued semantics; Introduction

Idea: Consider all the Boolean-valued subsets of the form

$$f : A \to \mathbb{B},$$

where A: the 1st-order universe, $\mathbb{B}$: a complete Boolean algebra

Boolean valued semantics; Introduction

Idea: Consider all the Boolean-valued subsets of the form

$$f : A \to \mathbb{B},$$

where A: the 1st-order universe, $\mathbb{B}$: a complete Boolean algebra

Note: When $\mathbb{B} = \{0, 1\}$, it is the same as considering all the subsets $\mathcal{P}(A)$, i.e., full semantics.

Boolean-valued semantics; Boolean-valued structures

From now on, $\mathcal{L}$ will be a relational language $\{R_1, \ldots, R_m\}$.

Boolean-valued semantics; Boolean-valued structures

From now on, $\mathcal{L}$ will be a relational language $\{R_1, \ldots, R_m\}$.

Definition

A Boolean-valued $\mathcal{L}$-structure is a tuple $M = (A, \mathbb{B}, \{R_i^M\})$ where

1. A is a nonempty set,
2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M : A^n \to \mathbb{B}$.

Boolean-valued semantics; Boolean-valued structures

From now on, $\mathcal{L}$ will be a relational language $\{R_1, \ldots, R_m\}$.

Definition

A Boolean-valued $\mathcal{L}$-structure is a tuple $M = (A, \mathbb{B}, \{R_i^M\})$ where

1. A is a nonempty set,
2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

Example

If $\mathbb{B} = \{0, 1\}$, R_i^M is a relation in 1st-order logic and M is the same as a 1st-order $\mathcal{L}$-structure.

Boolean valued semantics; the interpretation

From now on,

y or $\vec{y}$: 1st-order variables, X: 2nd-order variables.

Boolean valued semantics; the interpretation

From now on,

$$y \text{ or } \vec{y}\colon \text{ 1st-order variables, } X\colon \text{ 2nd-order variables.}$$

Definition

Let $M = (A, \mathbb{B}, \{R_i^M\})$ be a Boolean-valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M$ to each 2nd-order formula ϕ, $\vec{a} \in A^{<\omega}$, and $\vec{f} \in (\mathbb{B}^A)^{<\omega}$ as follows:

Boolean valued semantics; the interpretation

From now on,

y or $\vec{y}$: 1st-order variables, X: 2nd-order variables.

Definition

Let $M = (A, \mathbb{B}, \{R_i^M\})$ be a Boolean-valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M$ to each 2nd-order formula ϕ, $\vec{a} \in A^{<\omega}$, and $\vec{f} \in (\mathbb{B}^A)^{<\omega}$ as follows:

1. ϕ is $R_i(\vec{y})$. Then $\|R_i(\vec{y})[\vec{a}]\|^M = R_i^M(\vec{a})$.
2. ϕ is $X(y)$. Then $\|X(y)[a, f]\|^M = f(a)$.
3. Boolean combinations are as usual.
4. ϕ is $\exists y \psi$. Then $\|\exists y \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{b \in A} \|\psi[b, \vec{a}, \vec{f}]\|^M$.
5. ϕ is $\exists X \psi$. Then $\|\exists X \psi[\vec{a}, \vec{f}]\|^M = \bigvee_{g:\, A \to \mathbb{B}} \|\psi[\vec{a}, g, \vec{f}]\|^M$.

Boolean-valued semantics; the interpretation ctd.

Let $\mathbb{B}$ be a complete Boolean algebra. Then

$$(\mathbb{B}\text{-valued semantics}) = (\text{Full semantics in } V^{\mathbb{B}})$$

Boolean-valued semantics; the interpretation ctd.

Let $\mathbb{B}$ be a complete Boolean algebra. Then

$$(\mathbb{B}\text{-valued semantics}) = (\text{Full semantics in } V^{\mathbb{B}})$$

Definition

Let $M = (A, \mathbb{B}, \{R_i^M\})$ be a Boolean valued $\mathcal{L}$-structure and G be a $\mathbb{B}$-generic filter over V. Then for each n-ary predicate R_i, set $R_i^{M/G}$ as follows:

$$R_i^{M/G} = \{\vec{x} \in A^n \mid R_i^M(\vec{x}) \in G\}.$$

Then let M/G be the full 2nd-order $\mathcal{L}$-structure $(A, \mathcal{P}^{V[G]}(A), \{R_i^{M/G}\})$ in $V[G]$.

Boolean-valued semantics; the interpretation ctd..

$$(\mathbb{B}\text{-valued semantics}) = (\text{Full semantics in } V^{\mathbb{B}})$$

Key Lemma (Forcing Theorem)

If ϕ is a 2nd-order $\mathcal{L}$-sentence and $b \in \mathbb{B}$, then the following are equivalent:

1. $\|\phi\|^M = b$, and
2. for any $\mathbb{B}$-generic filter G over V with $b \in G$, $M/G \vDash \phi$.

Boolean-valued semantics; Validity

Definition

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean-valued $\mathcal{L}$-structure M.

Boolean-valued semantics; Validity

Definition

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean-valued $\mathcal{L}$-structure M.

Remark

ϕ is Boolean-valid if and only if ϕ is valid w.r.t. full semantics in any set generic extension.

Boolean-valued semantics; Validity

Definition

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean-valued $\mathcal{L}$-structure M.

Remark

ϕ is Boolean-valid if and only if ϕ is valid w.r.t. full semantics in any set generic extension.

Definition

$$0^{2f} = \{\phi \mid \phi \text{ is valid w.r.t. full semantics}\}$$
$$0^{2b} = \{\phi \mid \phi \text{ is Boolean valid}\}.$$

Boolean-valued semantics; Validity

Definition

A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean-valued $\mathcal{L}$-structure M.

Remark

ϕ is Boolean-valid if and only if ϕ is valid w.r.t. full semantics in any set generic extension.

Definition

$$0^{2f} = \{\phi \mid \phi \text{ is valid w.r.t. full semantics}\}$$
$$0^{2b} = \{\phi \mid \phi \text{ is Boolean valid}\}.$$

Question

0^{2f} or 0^{2b}, which is more complicated?

BV semantics vs full semantics; Validity

Theorem (Väänänen, I.)

If $V = \mathrm{L}$, then $0^{2b} \equiv_{\mathsf{T}} 0^{2f}$.

Point: Gödel's L is absolute in any set generic extension of L.

BV semantics vs full semantics; Validity

Theorem (Väänänen, I.)

If $V = \mathrm{L}$, then $0^{2b} \equiv_{\mathrm{T}} 0^{2f}$.

Point: Gödel's L is absolute in any set generic extension of L.

Theorem (Väänänen, I.)

If you assume the existence of proper class many Woodin cardinals and the Ω-Conjecture, then 0^{2b} is strictly simpler than 0^{2f}.

Point: $0^{2b} \equiv_{\mathrm{T}} 0^{\Omega} = \{\psi \mid \psi \text{ is } \Omega\text{-valid}\}$.

Compactness numbers

Definition

Let $\mathbb{L}$ be a logic. Then the Compactness number $\kappa^{\mathbb{L}}$ of $\mathbb{L}$ is defined as follows:

$$\kappa^{\mathbb{L}} = \min\{\kappa \mid \text{for any set } T \text{ of sentences in } \mathbb{L}\text{, if every subset of } T \text{ of size less than } \kappa \text{ has a model, then } T \text{ has a model}\}$$

Compactness numbers

Definition

Let $\mathbb{L}$ be a logic. Then the Compactness number $\kappa^{\mathbb{L}}$ of $\mathbb{L}$ is defined as follows:

$$\kappa^{\mathbb{L}} = \min\{\kappa \mid \text{for any set } T \text{ of sentences in } \mathbb{L}\text{, if every subset of } T \text{ of size less than } \kappa \text{ has a model, then } T \text{ has a model}\}$$

Example

If $\mathbb{L}$ is 1st-order logic with the standard semantics, then $\kappa^{\mathbb{L}} = \omega$.

BV semantics vs full semantics; Compactness

$\kappa^{2f} = \kappa^{\mathbb{L}}$ for $\mathbb{L} =$ SOL w.r.t. full semantics

$\kappa^{2b} = \kappa^{\mathbb{L}}$ for $\mathbb{L} =$ SOL w.r.t. Boolean-valued semantics

BV semantics vs full semantics; Compactness

$\kappa^{2f} = \kappa^{\mathbb{L}}$ for $\mathbb{L} =$ SOL w.r.t. full semantics

$\kappa^{2b} = \kappa^{\mathbb{L}}$ for $\mathbb{L} =$ SOL w.r.t. Boolean-valued semantics

Theorem (Magidor)

1. κ^{2f} exists if and only if there is an extendible cardinal.
2. If κ^{2f} exists, then κ^{2f} is the *least* extendible cardinal.

Definition

A cardinal κ is extendible if for any α with $\kappa < \alpha$, there are a β and an elementary embedding $j\colon (V_\alpha, \in) \to (V_\beta, \in)$ with critical point κ such that $j(\kappa) > \alpha$.

BV semantics vs full semantics; Compactness ctd.

Theorem (Väänänen, I.)

1. κ^{2b} exists if and only if there is a generically extendible cardinal.
2. If κ^{2b} exists, then κ^{2b} is the *least* generically extendible cardinal.

Definition

A cardinal κ is generically extendible if for any α with $\kappa < \alpha$, there is a set generic extension $V[G]$ of V such that in $V[G]$, there are a β and an elementary embedding $j\colon (V_\alpha, \in)^V \to (V_\beta, \in)^{V[G]}$ with critical point κ such that $j(\kappa) > \alpha$.

BV semantics vs full semantics; Compactness ctd..

Question

How small could generically extendible cardinals be?

BV semantics vs full semantics; Compactness ctd..

Question

How small could generically extendible cardinals be?

Theorem (Väänänen, I.)

Suppose there are proper class many Woodin cardinals. Then every successor cardinal is generically extendible. In particular, $\kappa^{2b} = \omega_1$.

Point: Use Stationary Tower Forcing $\mathbb{P}_{<\delta}$.

BV semantics vs full semantics; Compactness ctd..

Question

How small could generically extendible cardinals be?

Theorem (Väänänen, I.)

Suppose there are proper class many Woodin cardinals. Then every successor cardinal is generically extendible. In particular, $\kappa^{2b} = \omega_1$.

Point: Use Stationary Tower Forcing $\mathbb{P}_{<\delta}$.

Open problem

What is the consistency strength of a generically extendible cardinal?

Gödel's Constructible Hierarchy

Definition

$$
\begin{aligned}
\mathrm{L}_{0} &= \emptyset, \\
\mathrm{L}_{\alpha+1} &= \mathsf{Def}_{\mathsf{FOL}}\big((\mathrm{L}_{\alpha}, \in)\big), \\
\mathrm{L}_{\gamma} &= \bigcup_{\alpha<\gamma} \mathrm{L}_{\alpha} \qquad (\gamma \text{ is limit}), \\
\mathrm{L} &= \bigcup_{\alpha\in\mathsf{On}} \mathrm{L}_{\alpha}.
\end{aligned}
$$

Inner models from logics

Definition

Given a logic $\mathbb{L}$ extending FOL,

$$
\begin{aligned}
\mathrm{L}_0^{\mathbb{L}} &= \emptyset, \\
\mathrm{L}_{\alpha+1}^{\mathbb{L}} &= \mathrm{Def}_{\mathbb{L}}\big((\mathrm{L}_\alpha^{\mathbb{L}}, \in)\big), \\
\mathrm{L}_\gamma^{\mathbb{L}} &= \bigcup_{\alpha<\gamma} \mathrm{L}_\alpha^{\mathbb{L}} \qquad (\gamma \text{ is limit}), \\
\mathrm{L}(\mathbb{L}) &= \bigcup_{\alpha\in\mathrm{On}} \mathrm{L}_\alpha^{\mathbb{L}}.
\end{aligned}
$$

BV semantics vs full semantics; Gödel's L

Theorem (Myhill, Scott)

If $\mathbb{L}$ is SOL with full semantics, then $\mathrm{L}(\mathbb{L}) = \mathrm{HOD}$.

BV semantics vs full semantics; Gödel's L ctd.

Let L^{2b} be $\mathrm{L}(\mathbb{L})$ for $\mathbb{L} =$ SOL with Boolean-valued semantics.

Theorem (I.)

Suppose there are proper class many Woodin cardinals. Then L^{2b} is the least inner model M of ZFC such that for any poset P in M and any P-generic filter G over V,

$$(H_{\omega_1}, \in)^{M[G]} \prec (H_{\omega_1}, \in)^{V[G]}.$$

In particular, Projective Determinacy (PD) holds in L^{2b}.

BV semantics vs full semantics; Gödel's L ctd.

Let L^{2b} be $\mathrm{L}(\mathbb{L})$ for $\mathbb{L} = \mathsf{SOL}$ with Boolean-valued semantics.

Theorem (I.)

Suppose there are proper class many Woodin cardinals. Then L^{2b} is the least inner model M of ZFC such that for any poset P in M and any P-generic filter G over V,

$$(H_{\omega_1}, \in)^{M[G]} \prec (H_{\omega_1}, \in)^{V[G]}.$$

In particular, Projective Determinacy (PD) holds in L^{2b}. Also, L^{2b} is the least inner model of ZFC closed under the operator $(n, x) \mapsto M_n^{\#}(x)$. Furthermore,

1. L^{2b} is invariant under set forcings, and
2. L^{2b} is a premouse, hence is a model of GCH, $\diamondsuit_\kappa$, and $\square_\kappa$ for all uncountable regular cardinals κ.

Conclusion

- Henkin semantics and full semantics are the major semantics for 2nd-order logic.
- Henkin semantics is essentially the same as 1st-order logic.
- Full semantics is much more powerful than 1st-order logic and highly complicated.
- Boolean-valued semantics is a powerful semantics for 2nd-order logic while it could be simpler than full semantics under the existence of large cardinals.

The End.