Preserving sharps by tree forcings

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Infinite Combinatorics and Forcing Theory Workshop
Kyoto, December 1st

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- Large cardinals are strong axioms of infinity. Their existence, in each case, is equivalent to the existence of some especial elementary embedding.

Motivation: Large cardinals and forcing

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Question: Do small forcing notions preserve consequences of large cardinals in the same way?

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More generally, if x is a set of ordinals we say that $x^{\#}$ exists iff there is a non-trivial elementary embedding $j : L[x] \to L[x]$ that does no move ordinals up to $\sup(x)$.

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Theorem (folklore)

The property "For every set of ordinals x, $x^\#$ exists" is preserved by any forcing.

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It is consistent that every real has a sharp and there is a Σ^1_3-c.c.c. forcing notion such that in the generic extension holds $V = L[x]$ for some real x.

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However, if we impose some conditions over the forcing notion some positive results hold:

Theorem (Schlicht)

Suppose that $\mathbb{P} \subseteq \mathbb{R}$ is a provably Σ^1_2-definable c.c.c. forcing notion. Then, $\mathbb{P}$ preserves the property "every real has a sharp".

Our results

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Theorem (C.-Schlicht)

Suppose $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{U}, \mathbb{L}, \mathbb{ML}\}$. Then, $\mathbb{P}$ preserves the property "$x^\#$ exists for every real x". Furthermore, if $M_n^\#(x)$ exists for all $x \in \mathbb{R}$, $n \in \omega$, then the same holds in the generic extension. Equivalently, projective determinacy is preserved by $\mathbb{P}$.

Trees

Let $(T, <)$ be a tree and for every $t \in T$, let $\mathrm{Succ}_T(t)$ be the set of all immediate successors of $t \in T$.

Let $\mathrm{Split}(T) = \{t \in T : |\mathrm{Succ}_T(t)| > 1\}$ and let $\mathrm{Stem}(T)$ be the first element of $\mathrm{Split}(T)$.

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We say that T is perfect if for every $s \in T$ there exists $t \in \mathrm{Split}(T)$ such that $s < t$.

Arboreal forcing notions

Definition

A partial order $\mathbb{P}$ is arboreal if its conditions are perfect trees on ω or 2 ordered by inclusion. A partial order $\mathbb{P}$ is strongly arboreal if it is arboreal and for all $T \in \mathbb{P}$, if $t \in T$, $T_t = \{s \in T : \text{either } s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$.

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If $\mathbb{P}$ is strongly arboreal, we can code generic objects by reals in the standard way: if G is $\mathbb{P}$-generic over V, then $x_G = \bigcup\{\mathrm{Stem}(T) : T \in G\} = \bigcap\{[T] : T \in G\}$ is a real and $G = \{T \in \mathbb{P} : x_G \in [T]\}$

Sacks forcing

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Definition

Suppose that $S \in \mathbb{S}$. We define:

$$\mathbb{A}(\mathbb{S})_S = \{t \subseteq S : t \text{ is a finite subtree of } S \text{ isomorphic to some } {}^n2\}$$

ordered by end-extension, i.e. $t \leq s$ if and only if $t \supseteq s$ and $t{\restriction}_{|s|} = s$.

Given $S \in \mathbb{S}$, let $\pi_S : \mathrm{Split}(S) \to {}^{<\omega}2$ be the natural order isomorphism.

Lemma

Suppose that G is $\mathbb{A}(\mathbb{S})_S$-generic over V. Then:

- $T_G = \bigcup G$ is a perfect subtree of S.
- For every $x \in [T_G]$, $\pi_S(x) := \bigcup_{n<\omega} \pi_S(x\restriction_n)$ is Cohen-generic over V.

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Lemma (C.-S.)

Suppose that $\dot{x}$ is a name for a $\mathbb{S}$-generic real. Let $\sigma \in \mathrm{H}(\omega_1)$ be an $\mathbb{S}$-name for a real and let $S \in \mathbb{S}$. Suppose that M is an inner model with $\sigma, S \in M$ and $|\wp(\mathbb{A}(\mathbb{S})_S)^M| < \omega_1$. Then, there is some $T \leq S$ such that

$$T \Vdash_{\mathbb{S}} \dot{x} \textit{ is } \mathbb{C}\textit{-generic over } M \textit{ modulo } \pi_S$$

Lemma (C.-S.)

Suppose that V is closed under sharps for reals. Suppose that $r \in \mathbb{R}$. Then, for every $\mathbb{S}$-generic real x over V, there exists some real $y \in V$ such that x is $\mathbb{C}$-generic over $L[r, y]$.

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Proof.

Suppose $\dot{x}$ is a $\mathbb{S}$-name for x. As $(r, S)^{\#}$ exists, by the previous lemma applied to the model $L[r, S]$, the set

$$D = \{T \in \mathbb{S} : \text{for some } S \in \mathbb{S},\ T \leq S,\ T \Vdash_{\mathbb{S}} \dot{x} \text{ is } \mathbb{C}\text{-generic over } L[r, S]\}$$

is dense in $\mathbb{S}$.

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is dense in $\mathbb{S}$. If G is $\mathbb{S}$-generic over V containing $T \in D$, we can pick $S \geq T$ with

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$$T \Vdash_{\mathbb{S}} \dot{x} \text{ is } \mathbb{C}\text{-generic over } L[r, S] \text{ modulo } \pi_S$$

Therefore, $V[G] \models x$ is $\mathbb{C}$-generic over $L[r, S]$ modulo π_S. □

Theorem (C.-S.)

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Lift this embedding to the Cohen extension:

$$\begin{aligned} j' : L[r][x] &\to L[r][x] \\ \tau^x &\to (j(\tau))^x \end{aligned}$$

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Note that j' is elementary and non-trivial. Then $\bar{j} := j' \restriction_{L[y]}$ witness the existence of $y^{\#}$ in $V[x]$. □

Silver forcing

A Silver tree (or uniform tree) is a perfect tree $T \subseteq {}^{<\omega}2$ such that for all $s, t \in T$ of the same length we have

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Suppose that $S \subseteq^\omega 2$ is an uniform tree. We define

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If $t \in \mathbb{A}(\mathbb{U})_S$, let $\text{ht}(t)$ be the length of any branch through t and let $t{\restriction}_n = \{p \in t : |p| \leq n\}$.

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Proposition

Suppose that G is $\mathbb{A}(\mathbb{U})_S$ generic over some model V. Then $T_G = \bigcup G$ is a Silver subtree of S whose branches are $\mathbb{C}$-generic over V modulo π_S.

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Lemma (C.-S.)

Suppose that $\forall x \in \mathbb{R}(x^\#$ exists) and let $\sigma \in \mathrm{H}(\omega_1)$. Let $\dot{y}$ a name for a Silver real. For every $S \in \mathbb{U}$, there is some $T \leq S$ such that

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By proceeding in the same way that in Sacks forcing case (lifting the embedding to the Cohen extension) we have that Silver forcing also preserves sharps for reals.

Mathias forcing

Let $\mathbb{M} = \{(s, A) : s \in [\omega]^{<\omega}, A \in [\omega]^{\omega}, \max(s) < \min(A)\}$. In $\mathbb{M}$, we stipulate the following order:

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For each $(s, A) \in \mathbb{M}$, let

$$T_{(s,A)} := \{\chi_a^n{\restriction}_k : s \subseteq a \subseteq s \cup A,\ a \in [\omega]^{<\omega},\ k \leq n \in \omega\}$$

Let $\mathbb{M}_{\text{tree}} = \{T_{(s,A)} : (s, A) \in \mathbb{M}\}$. Then $\pi \colon \mathbb{M} \to \mathbb{M}_{\text{tree}}$ defined as $\pi(s, A) = T_{(s,A)}$ is an isomorphism.

Lemma (C.-S.)

Suppose that $\dot{G}$ is a name for a $\mathbb{M}$-generic filter. Let $p \in \mathbb{M}$ and suppose that M is an inner model with $\sigma, p \in M$ such that $|\wp(\mathbb{M})^M| < \omega_1$. Then, there is some $q \leq p$ such that

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Proceed as before...

Laver forcing

A tree $T \subseteq^{<\omega} [\omega]$ is called a *Laver tree* if it has a stem s and $\forall t \in T(t \subseteq s$ or $|\mathrm{Succ}_T(t)| = \omega)$.

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Note that $f(t)$ codes the values of t as blocks of 0 separated by 1.

Lemma

Let $S \in \mathbb{L}$, $\sigma \in \mathrm{H}(\omega_1)$ and consider $f : {}^{<\omega}\omega \to {}^{<\omega}2$ as above. Suppose that $u \subseteq \omega$ is Mathias generic over $L[\sigma, S]$. Then, $\bar{T} = f^{-1}[T_{(\varnothing,u)}]$ is a Laver tree with stem $\varnothing$.

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Suppose that V is closed under sharps for reals. Let $\dot{x}$ denote a name for a $\mathbb{L}$-generic real. Then for every $S \in \mathbb{L}$, $\sigma \in \mathrm{H}(\omega_1)$ there is some $T \leq S$ such that

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Remark: We proved also that the lemma above holds by replacing $\mathbb{S}$, $\mathbb{U}$ and $\mathbb{ML}$ instead of $\mathbb{L}$.

Other sharps

Let $n < \omega$, $x \in \mathbb{R}$. $M_n^{\#}(x)$ is the minimal countable active mouse constructed from x that contains n Woodin cardinals.

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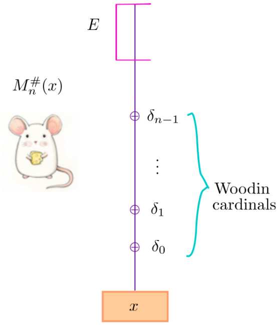

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By a result of Martin-Neeman-Steel-Woodin, projective determinacy is equivalent to the existence of $M_n^{\#}(x)$ for every $n < \omega$, $x \in \mathbb{R}$. We prove this theorem by induction on n and it is a little technical but we use strongly the suitable version of our star lemma.

Some applications

- If PD holds in V, forcing with some of the tree forcings above does not add new equivalence classes to thin equivalence relations on the reals.

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- If PD holds in V, forcing with some of the tree forcings above does not add new equivalence classes to thin equivalence relations on the reals.
- Suppose every real has a sharp. If $\mathbb{P}$ is some of the tree forcing notions mentioned then the value of the second uniform indiscernible

$$u_2 = \sup\{(\omega_1)^{+L[x]} \mid x \in {}^{\omega}2\}$$

does not change, i.e. $u_2^V = u_2^{V^{\mathbb{P}}}$.

Thank you for your attention!