Towers and Ultrafilters

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Disclaimer

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The views presented here are my own and do not necessarily reflect those of my coauthors.

Motivation

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Definition A **tower** is a descending sequence in $([\omega]^\omega, \subseteq^*)$ with no lower bound.

Observation Assume CH. Then every P-point is generated by a tower.

Question

- Can the assumption of being a P-point be weakened/dropped?
- Can CH be weakened/dropped?

Putting towers into ultrafilters

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Observation   If an ultrafilter is generated by a tower, then it is a P-point.

Question   Can every ultrafilter contain a tower?

A naive attempt

- Enumerate $[\omega]^\omega$ as $\{X_\alpha : \alpha < \mathfrak{c}\}$
- start with some $T_0 \in \mathcal{U}$ and recursively construct $\subseteq^*$-descending $T_\alpha \in \mathcal{U}$; at step α let $T_{\alpha+1}$ be either $T_\alpha \cap X_\alpha$ or $T_\alpha \setminus X_\alpha$, depending on which is in $\mathcal{U}$
- What to do at limits?
- It gets even worse: It can happen, that $\mathcal{U}$ contains a tower but at the same time has a basis such that no infinite sequence in the basis has a pseudointersection in $\mathcal{U}$. (For all we know, all ultrafilters can have such a basis!)
- Under CH it fails (Kunen, van Mill, Mills)!

RTFM

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Balcar, Frankiewicz, Mills: More on nowhere dense closed P-sets, Bull. Acad. Polon. Sci. Sér. Sci. Math, 28 (1980)

- the paper is published in an obscure journal, not available online
- need to know topology

Theorem It is consistent that ω^* can be covered by nowhere dense closed P-sets.

Translation It is consistent that every ultrafilter contains a tower.

Pull out the big guns

(an alternative proof)

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Observation There always is some ultrafilter, which contains a tower!

Make all ultrafilters look alike!

Definition (Blass) Two ultrafilters $\mathcal{V}_0, \mathcal{V}_1$ are nearly coherent if there is an ultrafilter $\mathcal{U}$ which is RB-below both of them.

Theorem (Blass) It is consistent that every two nonprincipal ultrafilters are nearly coherent.

Transfering towers

(under NCF)

Observation If $\mathcal{U}$ contains a tower than all ultrafilters RB-above $\mathcal{U}$ also contain towers.

Theorem If a tower $\mathcal{T}$ of length $\mathfrak{b}$ generates a nonmeager filter and $\mathcal{V}$ is an ultrafilter containing $\mathcal{T}$, then any ultrafilter $\mathcal{R}$B-below contains a tower.

Theorem (Solomon,Simon) If $\mathfrak{b} < \mathfrak{d}$ then there is a tower of length $\mathfrak{b}$ which generates a non-meager filter.

Note Under NCF every tower of length $\mathfrak{c}$ is meager! (It is not clear, whether there are towers of length $\mathfrak{c}$.)

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Question Assume MA. Does every P-point contain a tower?

Question Does $\mathfrak{d} = \mathfrak{c}$ imply that every P-point contains a tower?

Theorem Let $\omega < \lambda \leq \kappa$ be regular cardinals. Then it is consistent that

- $\mathfrak{t} = \lambda \leq \kappa = \mathfrak{c}$; and
- There is a selective ultrafilter which contains no tower.

In particular, it is consistent that $\mathfrak{d} = \mathfrak{c}$ and there is a P-point not containing a tower.

Constructing P-points without towers

- Assume $\mathfrak{t} = \mathfrak{c}$ and $\diamondsuit(S^{\mathfrak{c}}_{\geq\omega_1})$
- Recursively construct the ultrafilter in $\mathfrak{c}$ many steps
- Guarantee the following condition: there is a family $\{X_\alpha : \alpha < \mathfrak{c}\}$, such that if $X \in \mathcal{U}$ then $X_\alpha \subseteq X$ for all but boundedly many α.
- To get $\mathfrak{t} < \mathfrak{c}$, use a trick of Shelah:
  - Start with GCH and let $B = (\lambda^{<\lambda}, \supseteq)$.
  - Force $t = \mathfrak{c} = \kappa$, then force the diamond.
  - Finally, force with B^V.
  - This does not add reals and adds a tower of length λ thus making $\mathfrak{t} = \lambda$.
  - Finally, show that B^V does not add long towers into the ultrafilter.