Infinite Combinatorics, Definability, and Forcing

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Infinite Combinatorics and Forcing Theory

Discrete sets

Let R be a binary relation on a set X.

Definition

We say a set $A \subseteq X$ is **discrete** (w.r.t. R) $\iff$ no two distinct elements x, y of A are R-related.

Definition

We call such a set **maximal discrete** (w.r.t. R; short R-m.d.) if it is not a proper subset of any discrete set.

A is maximal discrete *iff* A is discrete and for any $x \in X \setminus A$

$$(\exists a \in A)\ (a\, R\, x) \vee (x\, R\, a)$$

Discrete sets (non-binary)

Let X be a set and $R \subseteq X^{<\omega}$.

Definition

We say a set $A \subseteq X$ is **discrete** (w.r.t. R) $\iff$ $A^{<\omega} \cap R = \emptyset$.

The notion of *maximal discrete set* is defined as before.

A is maximal discrete *iff* A is discrete and for any $x \in X \setminus A$

$$(\exists y_0, \ldots, y_n \in A \cup \{x\})\,(y_0, \ldots, y_n) \in R.$$

While maximal discrete sets always exist (under AC), one can study under which circumstances they can be *definable*.

Examples

There are many interesting examples where X is an effective Polish space and R is Borel:

Binary

- Transversals for equivalence relations
- mad families
- maximal eventually different families
- maximal orthogonal families of measures

Higher arity

- Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$
- Cofinitary groups

. . . and many more.

Example 1: mad families

Let $X = [\omega]^\omega$.

Define $R \subseteq X^2$ as follows: For $x, y \in X$,

$$x \, R \, y \iff x \cap y \text{ is infinite.}$$

A mad family is an infinite maximal discrete set w.r.t. R.

Some facts about mad families

1.
   - There is no analytic mad family (Mathias, 1968).
   - There is a (lightface) Π^1_1 mad family in $\mathbf{L}$ (A. R. Miller, 1987).
   - If there is a Σ^1_2 mad family, there is a Π^1_1 mad family (Törnquist, 2013).
2.
   - One can find mad families which remain mad after forcing (for various forcings; many authors).
   - In particular, the existence of a Π^1_1 mad family is consistent with arbitrary values for 2^ω.
3. One can force that there are no definable mad families:
   - Mathias, ca. 1969: from a Mahlo,
   - Törnquist, 2015: from an inaccessible (no mad families in Solovay's model),
   - Horowitz-Shelah, 2016: from ZFC.

“there is no projective R-m.d. family” is equiconsistent with ZFC in several other cases, as well:

- maximal eventually different families of functions (Brendle-Khomskii, unpublished)
- maximal orthogonal families of measures (Fischer-Törnquist, 2010); This is because the same holds for “every projective set has the Baire property”

Example 2

The statement that there are no definable R-m.d. sets can have large cardinal strength:

Theorem (Horowitz-Shelah, 2016)

There is a Borel binary relation R on 2^ω (in fact, a graph relation) such that “there is no projective R-m.d. set” is equiconsistent with the existence of an inaccessible cardinal.

Example 3

Let $X = \omega^\omega$ and for $f, g \in X$ let

$$f \mathrel{R} g \iff \{n \mid f(n) = g(n)\} \text{ is infinite.}$$

A (maximal) discrete set w.r.t. R is a (maximal) *eventually different family*.

Theorem (Horowitz-Shelah, 2016)

(ZF) There is a Borel maximal eventually different family.

Such a family remains m.d. in any larger universe.

Example 4: Orthogonality of measures

- Let $P(2^\omega)$ be the set of Borel probability measures on 2^ω.
- Two measures $\mu, \nu \in P(2^\omega)$ are said to be orthogonal, written

$$\mu \perp \nu$$

exactly if: there is a Borel set $A \subseteq 2^\omega$ such that

$$\mu(A) = 1$$

and

$$\nu(A) = 0.$$

- Note that $P(2^\omega)$ is an effective Polish space.

History of maximal orthogonal families

Question (Mauldin, circa 1980)

Can a **mof** in $P(2^\omega)$ be analytic?

The answer turned out to be ‘no’:

Theorem (Preiss-Rataj, 1985)

*There is no analytic **mof** in $P(2^\omega)$.*

This is optimal, in a sense:

Theorem (Fischer-Törnqust, 2009)

*In $\mathbf{L}$, there is a Π^1_1 **mof** in $P(2^\omega)$.*

In fact:

Theorem

*If there is a Σ^1_2 **mof** in $P(2^\omega)$, there is a Π^1_1 **mof**.*

Mofs and forcing

Mofs are fragile creatures:

Facts

1. Adding any real destroys maximality of **mofs** from the groundmodel (observed by Ben Miller; not restricted to forcing extensions)
2. If there is a Cohen real over **L**, there are no Σ^1_2 **mofs** in $P(2^\omega)$ (F-T, 2009)
3. The same holds if there is a random real over **L** (Fischer-Friedman-Törnquist, 2010).
4. The same holds if there is a Mathias real over **L** (S-Törnquist, 2015).

Question (F-T, 2009)

If there is a Π^1_1 **mof**, does it follow that $\mathcal{P}(\omega) \subseteq \mathbf{L}$?

Π^1_1 mofs in extensions of $\mathbf{L}$

Theorem (S-Törnquist, 2015)

*If s is Sacks over $\mathbf{L}$ there is a (lightface!) Π^1_1 **mof** in $\mathbf{L}[s]$.*

Theorem (S 2016)

*The statement 'there is a Π^1_1 **mof** ' is consistent with $2^\omega = \omega_2$.*

In fact :

Theorem (S 2016)

Let R be a binary Σ^1_1 relation on an effective Polish space X. If $\bar{s}$ is generic for iterated Sacks forcing over $\mathbf{L}$, there is a (lightface) Δ^1_2 maximal $\mathcal{R}$-discrete set in $\mathbf{L}[\bar{s}]$.

Here is the main idea of the proof in the case of adding a single Sacks real.

Assume R is symmetric (otherwise, look at $R \cup R^{-1}$).

Of course Sacks forcing $\mathbb{S}$ is the set of *perfect trees* $p \subseteq 2^{<\omega}$, ordered by inclusion and $[p]$ is the set of branches through p.

We need the following theorem of Galvin:

Theorem (Galvin's Theorem)

Let $p \in \mathbb{S}$ and

$$c\colon [p]^2 \to \{0,1\}$$

be symmetric and Baire measurable.
Then there is $q \in \mathbb{S}$, $q \leq p$ such that c is constant on

$$[q]^2 \setminus \mathrm{diag}$$

The m.d. set will be a union of ω_1-many perfect sets:
For some for sequence of perfect sets $\langle P_\xi \mid \xi < \omega_1 \rangle \in \mathbf{L}$, our m.d. set is just

$$\bigcup_{\xi<\omega_1} \bar{P}_\xi^{\mathbf{L}[s]}.$$

We construct this sequence by induction: Say we have $\langle P_\nu \mid \nu < \xi \rangle \in \mathbf{L}$, and say we have $p \in \mathbb{S}$ and a $\mathbb{S}$-name which are candidates for

$$p \Vdash \dot{x} \text{ is not in our family.}$$

So assume that $p \Vdash (\forall y \in \bigcup_{\nu<\xi} \bar{P}_\nu)\neg(\dot{x} \, R \, y)$.
We can also assume $\dot{x}$ is given by a continuous function $f\colon 2^\omega \to X$:

$$p \Vdash \dot{x} = f(s_{\dot{G}}).$$

It is easy to thin out p so that

$$(\forall x \in f''[p])(\forall y \in \bigcup_{\nu<\xi} P_\nu)\neg(x \, R \, y).$$

Further thin out such that either:

1. $f''[p]$ is R-discrete
2. $f''[p]$ is R-complete, i.e. $(\forall x, y \in f''[p])\ x\, R\, y$.

In the first case, let $P_\xi = f''[p]$. As

$$p \Vdash \dot{x} = f(s_{\dot{G}}) \in \bar{P}_\xi$$

we have dealt with this candidate.
In the second case, let $P_\xi = \{f(z)\}$, where z is the left-most branch through p. By Shoenfield absoluteness

$$p \Vdash \dot{x} = f(s_{\dot{G}})\ R\ f(z),$$

so we have again dealt with this candidate.
In either case, $\bigcup_{\nu \leq \xi} P_\nu$ is discrete. By Π^1_1 absoluteness, this will hold for $\bigcup_{\xi < \omega_1} \bar{P}_\xi^{\mathbf{L}[s]}$ as well.
As there are only ω_1-many pairs $(\dot{x}, p)$ as above, we can ensure maximality.

A basis for $\mathbb{R}$ over $\mathbb{Q}$

Let $X = \mathbb{R}$ and let R be the set of finite tuples from X which are linearly dependent over $\mathbb{Q}$.

A more involved proof but using similar ideas as in the previous sketch (including a generalization of Galvin's theorem to k-tuples due to Blass) gives us:

Theorem (S 2016)

If s is a Sacks real over $\mathbf{L}$, there is a Π^1_1 basis for $\mathbb{R}$ over $\mathbb{Q}$ in $\mathbf{L}[s]$.

What is Galvin's Theorem for iterated Sacks forcing?

- Let $\bar{P}$ be iterated Sacks forcing and $\bar{p} \in \mathbb{P}$. What is $[\bar{p}]$?
- Provided we can define $[\bar{p}]$...

Question:

Is there for every $\bar{p} \in \mathbb{P}$ and every

$$c\colon [\bar{p}]^2 \to \{0,1\}$$

which is symmetric and *nice*, some $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c is constant on $[\bar{q}]^2 \setminus \text{diag}$?

What do I mean by *nice*?

- Answer is 'yes' for c which is *continuous* on $[\bar{p}]^2 \setminus \text{diag}$ (Geschke-Kojman-Kubiś-Schipperus)
- perhaps Baire measurable...?

For a dense set of $\bar{p} \in \mathbb{P}$ we have:

0 There is $F_0 \colon [\bar{p}(0)] \to$ PERFECT TREES and $\sigma_1 \in \operatorname{supp}(\bar{p})$ such that

$$\bar{p} \Vdash \bar{p}(\sigma_1) = F_0(\bar{s}_{\dot{G}}(0))$$

1 There is a continuous function F_1 and $\sigma_2 \in \operatorname{supp}(\bar{p})$ such that (letting $\sigma_0 = 0$)

$$\bar{p} \Vdash \bar{p}(\sigma_2) = F_1(\bar{s}_{\dot{G}} \restriction \sigma_2)$$

ω And so on: There exists sequences $F_0, \ldots, F_k, \ldots$ and $\sigma_0, \ldots, \sigma_k, \ldots$ with $\sigma_0 = 0$ such that the analogous holds for each $k \in \omega$ and

$$\{\sigma_k \mid k \in \omega\} = \operatorname{supp}(\bar{p})$$

Then $[\bar{p}]$ is the subspace of $(2^\omega)^\lambda$ consisting of

$$\bar{x} \colon \operatorname{supp}(\bar{p}) \to 2^\omega$$

such that for each $n \in \omega$

$$\bar{x}(\sigma_n) \in [F_n(\bar{x} \restriction \sigma_n)]$$

A counterexample

Let $\bar{p} \in \mathbb{P}$. Fix $\xi < \lambda$.

Define a symmetric Borel function

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

by

$$c(\bar{x}_0, \bar{x}_1) = \begin{cases} 1 & \text{if } \bar{x}_0(\xi) \neq \bar{x}_1(\xi) \\ 0 & \text{otherwise} \end{cases}$$

Note:

- Every $\bar{q} \leq \bar{p}$ will meet both colours
- $c^{-1}(1)$ is open, $c^{-1}(0)$ is closed.

For $\bar{x}_0, \bar{x}_1 \in [\bar{p}]$, let

$$\Delta(\bar{x}_0, \bar{x}_1) = \text{the least } \xi \text{ such that } \bar{x}_0(\xi) \neq \bar{x}_1(\xi).$$

Let

$$\Delta_\xi = \{(\bar{x}_0, \bar{x}_1) \in [\bar{p}]^2 \mid \Delta(\bar{x}_0, \bar{x}_1) = \xi\}$$

- Δ_0 is comeager in $[\bar{p}]^2$
- So *nice* must be more restrictive than Baire measurable!
- otherwise: take c arbitrary on Δ_ξ, $\xi > 0$ (a meager set!)

Another counterexample:

Fix a bijection $G\colon \mathrm{supp}(\bar{p}) \to \omega$.
Define a symmetric function

$$c\colon [\bar{p}]^2 \to \{0,1\}$$

as follows:

Suppose we have $\bar{x}_0, \bar{x}_1 \in [\bar{p}]$ and suppose $\bar{x}_0 <_{\text{lex}} \bar{x}_1$. Let

$$\xi = \Delta(\bar{x}_0, \bar{x}_1).$$

If $\xi \in \mathrm{supp}(\bar{p})$ and $G(\xi) = k$, set

$$c(\bar{x}_0, \bar{x}_1) = \bar{x}_i(0)(k).$$

(When $\xi \in \mathrm{supp}(\bar{p})$ fails, set c to be 0; this case is irrelevant)

Now if $c(\cdot,\cdot)$ only depends on $\Delta(\cdot,\cdot)$ on some $[q]$, $[q(0)]$ can contain at most *two* branches, contradiction.

The solution:

Theorem (Galvin's Theorem for iterated Sacks forcing)

For every $\bar{p} \in \mathbb{P}$ and every symmetric universally Baire

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

there is $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$, with an enumeration $\langle \sigma_k \mid k \in \omega \rangle$ of $\operatorname{supp}(\bar{q})$ and a function $N\colon \operatorname{supp}(\bar{q}) \to \omega$ such that for $(\bar{x}_0, \bar{x}_1) \in [\bar{q}]^2 \setminus \mathrm{diag}$, the value of $c(\bar{x}_0, \bar{x}_1)$ only depends on

$$\xi = \Delta(\bar{x}_0, \bar{x}_1)$$

and the following (finite) piece of information:

$$(\bar{x}_0 \restriction K, \bar{x}_1 \restriction K)$$

where $K = \{\sigma_0, \ldots, \sigma_{N(\xi)}\} \times N(\xi)$.

Example 5: Cofinitary groups

- Work in the space S_∞, the group of bijections from $\mathbb{N}$ to itself (permutations).
- $\mathrm{id}_{\mathbb{N}}$ is the identity function on $\mathbb{N}$, the neutral element of S_∞.

Definition

We say $g \in S_\infty$ is *cofinitary* $\iff$

$$\{n \in \mathbb{N} \mid g(n) = n\} \text{ is finite.}$$

$\mathcal{G} \leq S_\infty$ is *cofinitary* $\iff$ every $g \in \mathcal{G} \setminus \{\mathrm{id}_{\mathbb{N}}\}$ is cofinitary.

Definability of mcgs

Theorem (Kastermans)

No mcg can be K_σ.

Some history:

- Gao-Zhang: If $\mathbf{V} = \mathbf{L}$, there is a mcg with a Π^1_1 set of generators.
- Kastermans: If $\mathbf{V} = \mathbf{L}$, there is a Π^1_1 mcg.
- Fischer-S.-Törnquist, 2015: The existence of a Π^1_1 mcg is consistent with arbitrarily large continuum.

Theorem (Horowitz-Shelah, 2016)

(ZF) There is a Borel maximal cofinitary group.

By $\mathbf{\Sigma}^1_2$ absoluteness, a Borel mcg remains maximal in any outer model.
They also claim they will show there is a closed mcg in a future paper.

Thank You!