On strong negation of FRP

Hiroshi Sakai

Kobe University

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Notation

Let λ be a regular cardinal $\geq \omega_2$.

- For a regular $\mu < \lambda$ let

$$E^\lambda_\mu := \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \mu\} .$$

- Suppose $S \subseteq E^\lambda_\omega$. A *ladder system* on S is a sequence $\vec{b} = \langle b_\alpha \mid \alpha \in S \rangle$ s.t. each b_α is an unbounded subset of α of order-type ω.

- For a ladder system $\vec{b} = \langle b_\alpha \mid \alpha \in S \rangle$ on $S \subseteq E^\lambda_\omega$ let

$$X_{\vec{b}} := \{x \in [\lambda]^\omega \mid \sup(x) \in S \ \&\ |x \cap b_{\sup(x)}| = \omega\} ,$$

$$Y_{\vec{b}} := \{x \in [\lambda]^\omega \mid \sup(x) \in S \ \&\ |x \cap b_{\sup(x)}| < \omega\} .$$

  Both $X_{\vec{b}}$ and $Y_{\vec{b}}$ are stationary in $[\lambda]^\omega$ if S is stationary in λ.

Fodor-type Reflection Principle (FRP)

Defenition (Fuchino-Juhász-Soukup-Szentmiklóssy-Usuba)

- For a regular cardinal $\lambda \geq \omega_2$,

  $\mathrm{FRP}(\lambda) \equiv$ for any stationary $S \subseteq E^\lambda_\omega$ and any ladder system $\vec{b}$ on S there are stationary many $\gamma \in E^\lambda_{\omega_1}$ such that $X_{\vec{b}} \cap [\gamma]^\omega$ is stationary in $[\gamma]^\omega$.

- $\mathrm{FRP} \equiv \mathrm{FRP}(\lambda)$ for all regular $\lambda \geq \omega_2$.

Theorem (F-J-S-S-U, Fuchino-Soukup-Sakai-Usuba)

FRP is equivalent to each of the following assertions:

1. For any locally compact topological space X, if X is non-metrizable, then there is a non-metrizable $Y \subseteq X$ of size ω_1.
2. If G is a graph with $\mathrm{col}(G) > \omega$, then there is $H \subseteq G$ of size ω_1 with $\mathrm{col}(H) > \omega$.

$\vdots$

Consystency of FRP

Theorem (F-J-S-S-U)

If a strongly compact cardinal is Lévy collapsed to ω_2, then FRP holds.

Theorem (F-J-S-S-U)

$\mathrm{FRP}(\lambda^+) \;\Rightarrow\; \neg\square_\lambda$.

Strong negation of FRP

Let λ be a regular cardinal $\geq \omega_2$.

$\neg$FRP(λ) $\Leftrightarrow$ There are a stationary $S \subseteq E^\lambda_\omega$ and a ladder system $\vec{b}$ on S s.t. there are club many $\gamma \in E^\lambda_{\omega_1}$ with $X_{\vec{b}} \cap [\gamma]^\omega$ nonstationary in $[\gamma]^\omega$.

Defenition (Strong Negation of FRP)

SNFRP(λ) $\equiv$ For any stationary $S \subseteq E^\lambda_\omega$ and any ladder system $\vec{b}$ on S there are club many $\gamma \in E^\lambda_{\omega_1}$ such that $X_{\vec{b}} \cap [\gamma]^\omega$ is nonstationary in $[\gamma]^\omega$.

$\Leftrightarrow$ For any ladder system $\vec{b}$ on E^λ_ω there are club many $\gamma \in E^\lambda_{\omega_1}$ s.t. $X_{\vec{b}} \cap [\gamma]^\omega$ is nonstatioanry in $[\gamma]^\omega$.

$\Leftrightarrow$ For any ladder system $\vec{b}$ on E^λ_ω there are club many $\gamma \in E^\lambda_{\omega_1}$ s.t. $Y_{\vec{b}} \cap [\gamma]^\omega$ contains a club set in $[\gamma]^\omega$.

- SNFRP(λ) is a very strong reflection principle for $Y_{\vec{b}}$.

General Questions

1. Is SNFRP(λ) consistent?
2. Are there statements in topology or graph theory equivalent to SNFRP(λ)?

Inconsistency of SNFRP

Defenition (Partial square)

For an uncountable cardinal λ,

$\square^p_\lambda \equiv$ there are a stationary $T \subseteq E^{\lambda^+}_{\omega_1}$ and a sequence $\langle c_\gamma \mid \gamma \in T \rangle$ s.t.

- c_γ is a club subset of γ of order-type ω_1,
- if $\alpha \in \mathrm{Lim}(c_\gamma) \cap \mathrm{Lim}(c_\delta)$, then $c_\gamma \cap \alpha = c_\delta \cap \alpha$.

Fact (Shelah)

- $\square_\lambda$ implies $\square^p_\lambda$.
- (Shelah) $\square^p_\lambda$ holds for every regular cardinal $\lambda \geq \omega_2$.

Proposition 1

$\square^p_\lambda$ implies that SNFRP(λ^+) fails for any uncountable cardinal λ.
In particular, SNFRP(λ^+) fails for any regular $\lambda \geq \omega_2$.

Proof of Proposition 1

$\square^p_\lambda \equiv$ There are a stationary $T \subseteq E^{\lambda^+}_{\omega_1}$ and a sequence $\langle c_\gamma \mid \gamma \in T \rangle$ s.t.

- c_γ is a club subset of γ of order-type ω_1,
- if $\alpha \in \mathrm{Lim}(c_\gamma) \cap \mathrm{Lim}(c_\delta)$, then $c_\gamma \cap \alpha = c_\delta \cap \alpha$.

$\mathsf{SNFRP}(\lambda^+) \equiv$ For any ladder system $\vec{b}$ on $E^{\lambda^+}_\omega$ there are club many $\gamma \in E^{\lambda^+}_{\omega_1}$ s.t. $X_{\vec{b}} \cap [\gamma]^\omega$ is nonstationary in $[\gamma]^\omega$.

Proof of Proposition 1

- Let $S := \bigcup_{\gamma \in T} \mathrm{Lim}(c_\gamma)$.
  For each $\alpha \in S$, taking $\gamma \in T$ with $\alpha \in \mathrm{Lim}(c_\gamma)$, let $c_\alpha := c_\gamma \cap \alpha$.
- For each $\gamma \in T$ the following X_γ is club in $[\gamma]^\omega$:

$$X_\gamma := \{x \in [\gamma]^\omega \mid \sup(x) \in \mathrm{Lim}(c_\gamma) \ \& \ c_\gamma \cap \sup(x) \subseteq x\} .$$

  Moreover $X_\gamma \subseteq \{x \in [\gamma]^\omega \mid c_{\sup(x)} \subseteq x\}$.
- For each $\alpha \in S$ take an unbounded $b_\alpha \subseteq c_\alpha$ of order-type ω.
  Then $X_{\vec{b}} \cap [\gamma]^\omega$ is stationary for every $\gamma \in T$. □

Consistency of SNFRP

Prop. 1 leaves a possibility of that SNFRP(λ) is consistent for the following λ:

- ω_2
- successors of singular cardinals
- weakly inaccessible cardinals

Main Theorem

SNFRP(ω_2) is consistent.

Outline of consistency of SNFRP(ω_2)

For a ladder system $\vec{b}$ on $E^{\omega_2}_{\omega}$ let

$$T_{\vec{b}} := \{\gamma \in E^{\omega_2}_{\omega_1} \mid Y_{\vec{b}} \cap [\gamma]^{\omega} \text{ contains a club set in } [\gamma]^{\omega}\} .$$

Starting from a model of MM, we will iterate club shootings through $T_{\vec{b}} \cup E^{\omega_2}_{\omega}$ for all ladder system $\vec{b}$ on $E^{\omega_2}_{\omega}$:

- For a ladder system $\vec{b}$ on $E^{\omega_2}_{\omega}$ let $\mathbb{C}(\vec{b})$ be the poset of all bounded closed subsets of $T_{\vec{b}} \cup E^{\omega_2}_{\omega}$ ordered by reverse inclusions.
- $\mathbb{C}(\vec{b})$ is σ-closed and has size 2^{ω_1}. So if $2^{\omega_1} = \omega_2$, then $\mathbb{C}(\vec{b})$ preserves all cardinals except for ω_2.
- MM implies that $T_{\vec{b}}$ is stationary. So $\mathbb{C}(\vec{b})$ is $<\omega_2$-Baire (thus preserves ω_2).
- Moreover we can prove that, under MM, any ω_1-support iteration of $\mathbb{C}(\vec{b})$'s is $<\omega_2$-Baire.
- So, under MM, we can construct an ω_1-support iteration of $\mathbb{C}(\vec{b})$ for all $\vec{b}$ without collapsing any cardinal. □

Questions on consistency of SNFRP

Question

What is the consistency strength of SNFRP(ω_2)?

Question

1. Is SNFRP(λ) consistent for λ a successor cardinal of a singular cardinal?
2. Is SNFRP(λ) consistent for a weakly inaccessible cardinal λ?

Equivalent statement

I have not found any statement in topology or graph theory equivalent to SNFRP. But there is one on infinite combinatorics.

Defenition

Suppose $S \subseteq E^{\omega_2}_{\omega}$. A ladder system $\vec{b} = \langle b_\alpha \mid \alpha \in S \rangle$ is said to be *strongly almost disjoint* if for any $\gamma < \omega_2$ there is a regressive function f on $S \cap \gamma$ such that $\langle b_\alpha \setminus f(\alpha) \mid \alpha \in S \cap \gamma \rangle$ is pairwise disjoint.

Theorem (F-J-S-S-U)

The following are equivalent:

1. FRP(ω_2)
2. For any stationary $S \subseteq E^{\omega_2}_{\omega}$ and any ladder system $\vec{b}$ on S, $\vec{b}$ is not strongly almost disjoint.

Proposition 2

The following are equivalent:

1. SNFRP(ω_2).
2. For any ladder system $\langle b_\alpha \mid \alpha \in E^{\omega_2}_{\omega} \rangle$ there is a club $C \subseteq \omega_2$ such that $\langle b_\alpha \mid \alpha \in C \cap E^{\omega_2}_{\omega} \rangle$ is strongly almost disjoint.

Proof of (1) $\Rightarrow$ (2)

- Take an arbitrary ladder system $\vec{b} = \langle b_\alpha \mid \alpha \in E^{\omega_2}_\omega \rangle$.
  Let $C \subseteq \omega_2$ be a club such that $Y_{\vec{b}}$ contains a club set for all $\gamma \in C \cap E^{\omega_2}_{\omega_1}$.
  Let $S := C \cap E^{\omega_2}_\omega$. For each $\gamma \in E^{\omega_2}_{\omega_1}$ we will find a regressive function f on $S \cap \gamma$ with $\langle b_\alpha \setminus f(\alpha) \mid \alpha \in S \cap \gamma \rangle$ pairwise disjoint.
- Let Z be the set of all $x \in [\omega_2]^\omega$ such that
  - $b_\alpha \subseteq x$ for all $\alpha \in S \cap x$,
  - $|x \cap b_\alpha| < \omega$ for all $\alpha \in S \setminus x$.

  Then $Z \cap [\gamma]^\omega$ contains a club set for all $\gamma \in E^{\omega_2}_{\omega_1}$.
- Fix $\gamma \in E^{\omega_2}_{\omega_1}$. Let $\langle x_\xi \mid \xi < \omega_1 \rangle$ be a $\subseteq$-inc. cont. cof. sequence in $Z \cap [\gamma]^\omega$.
  Let ξ_α be such that $\alpha \in x_{\xi_\alpha + 1} \setminus x_{\xi_\alpha}$.
- Take a regressive function g on $S \cap \gamma$ with $(b_\alpha \setminus g(\alpha)) \cap x_{\xi_\alpha} = \emptyset$.
  Note that if $\xi_\alpha \neq \xi_\beta$, then $b_\alpha \setminus g(\alpha)$ and $b_\beta \setminus g(\beta)$ are disjoint.
- Because each x_ξ is countalbe, we can take a regressive function h on $S \cap \gamma$ such that if $\xi_\alpha = \xi_\beta$, then $b_\alpha \setminus h(\alpha)$ and $b_\beta \setminus h(\beta)$ are disjoint.
- Then $f(\alpha) := \max(g(\alpha), h(\alpha))$ is as desired. $\square$.

Some application

For $X \subseteq [\omega_2]^\omega$ we say that $\sup \restriction X$ is injective if $\sup(x) \neq \sup(y)$ for any distinct $x, y \in X$.

Open Problem

Is it consistent that there is no stationary $X \subseteq [\omega_2]^\omega$ with $\sup \restriction X$ injective.

We say that $X \subseteq [\omega_2]^\omega$ is *reflecting stationary* if there are stationary many $\gamma < \omega_2$ with $X \cap [\gamma]^\omega$ is stationary.

Proposition 3

SNFRP(ω_2) implies that there is no reflecting stationary $X \subseteq [\omega_2]^\omega$ with $\sup \restriction X$ injective.