Reflection theorems on non-existence of orthonormal bases of pre-Hilbert spaces

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Orthonormal bases of a pre-Hilbert space

- We fix $K = \mathbb{R}$ or $\mathbb{C}$ (all of the following arguments work for both of the scalar fields). In this talk, we work throughout in ZFC.
- An inner-product space over K is also called a **pre-Hilbert space** (over K).
- For a pre-Hilbert space with the inner product $(\mathbb{x}, \mathbb{y}) \in K$ for $\mathbb{x}$, $\mathbb{y} \in X$, $B \subseteq X$ is **orthonormal** if $(\mathbb{x}, \mathbb{x}) = 1$ and $(\mathbb{x}, \mathbb{y}) = 0$ for all distinct $\mathbb{x}$, $\mathbb{y} \in B$.
- $B \subseteq X$ is an **orthonormal basis** of X if B is orthonormal and spans a K-subspace of X which is dense in X.

If $B \subseteq X$ is an orthonormal basis of X then B is a maximal orthonormal system of X.

▷ If X is not complete the reverse implication is not necessary true!

Orthonormal bases of a pre-Hilbert space (2/2)

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Example 1. Let X be the sub-inner-product-space of $\ell_2(\omega+1)$ spanned by $\{\mathbb{e}_n^{\omega+1} : n \in \omega\} \cup \{\mathbb{b}\}$
where $\mathbb{b} \in \ell_2(\omega+1)$ is defined by

(1) $\mathbb{b}(\omega) = 1$;

(2) $\mathbb{b}(n) = \frac{1}{n+2}$ for $n \in \omega$.

Then $B = \{\mathbb{e}_n^{\omega+1} : n \in \omega\}$ is a maximal orthonormal system in X but it is not a basis of X.

Notation

- Note that X in Example 1 has an orthonormal basis.

Pre-Hilbert spaces without orthonormal bases

Lemma 2. (P. Halmos 196?) There are pre-Hilbert spaces X of dimension $\aleph_0$ and density λ for any $\aleph_0 < \lambda \leq 2^{\aleph_0}$.

Proof. Let B be a linear basis (Hamel basis) of the linear space $\ell_2(\omega)$ extending $\{\mathbb{e}_n^\omega : n \in \omega\}$. Note that $|\,B\,| = 2^{\aleph_0}$ (Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω of cardinality $2^{\aleph_0}$. For each $a \in \mathcal{A}$ let $\mathbb{b}_a \in \ell_2(\omega)$ be s.t. $\mathrm{supp}(\mathbb{b}_a) = a$. Then $\{\mathbb{b}_a : a \in \mathcal{A}\}$ is a linearly independent subset of $\ell_2(\omega)$ of cardinality $2^{\aleph_0}$).

Notation

Let $f : B \to \{\mathbb{e}_\alpha^\lambda : \alpha < \lambda\} \cup \{\mathbb{0}_{\ell_2(\lambda)}\}$ be a surjection s.t. $f(\mathbb{e}_n^\omega) = \mathbb{0}_{\ell_2(\lambda)}$ for all $n \in \omega$. Note that f generates a linear mapping from the linear space $\ell_2(\omega)$ to a dense subspace of $\ell_2(\lambda)$. Let $U = \{\langle \mathbb{b}, f(\mathbb{b})\rangle : \mathbb{b} \in B\}$ and $X = [U]_{\ell_2(\omega)\oplus\ell_2(\lambda)}$. Then this X is as desired since $\{\langle \mathbb{e}_n^\omega, \mathbb{0}\rangle : n \in \omega\}$ is a maximal orthonormal system in X while we have $\mathrm{cls}_{\ell_2(\omega)\oplus\ell_2(\lambda)}(X) = \ell_2(\omega) \oplus \ell_2(\lambda)$ and hence $d(X) = \lambda$. $\square$

Dimension and density of a pre-Hilbert space

- With practically the same proof, we can also show:

> **Lemma 3.** (A generalization of P. Halmos' Lemma) For any cardinal κ and λ with $\kappa < \lambda \leq \kappa^{\aleph_0}$, there are (pathological) pre-Hilbert spaces of dimension κ and density λ. $\square$

- The dimension and density of a pre-Hilbert space cannot be more far apart:

> **Proposition 4.** (D. Buhagiara, E. Chetcutib and H. Weber 2008) For any pre-Hilbert space X, we have $d(X) \leq |X| \leq (\dim(X))^{\aleph_0}$.

The proof of Proposition 4.

Pathological pre-Hiblert spaces

- We call a pre-Hilbert space X without any orthonormal bases **pathological**.

- If X is pathological then $d(X) > \aleph_0$
(if $d(X) = \aleph_0$ we can construct an orthonormal basis by Gram-Schmidt process).

- There are also pathological pre-Hilbert spaces X with $\dim(X) = d(X) = \kappa$ for all uncountable κ (see Corollary 7 on the next slide).
- ▷ Thus there are non-separable pre-Hilbert spaces without orthonormal basis in all possible combination of dimension and density.

Characterization of pathology

Lemma 5. Suppose that X is a pre-Hilbert space with an orthonormal basis (i.e. non-pathological) and X is a dense linear subspace of $\ell_2(\kappa)$. If χ is a large enough regular cardinal, and $M \prec \mathcal{H}(\chi)$ is s.t. $X \in M$ then $X = X \downarrow (\kappa \cap M) \oplus X \downarrow (\kappa \setminus M)$.

Notation

Theorem 6. Suppose that X is a pre-Hilbert space and X is a dense linear subspace of $\ell_2(S)$. Then X is non-pathological if and only if there is a partition $\mathcal{P} \subseteq [S]^{\leq\aleph_0}$ of S s.t. $X = \overline{\bigoplus}_{A\in\mathcal{P}} X \downarrow A$.

Proof. For $\Rightarrow$ use Lemma 5 (with countable M's) repeatedly. $\square$

Corollary 7. Suppose that X and Y are pre-Hilbert spaces if one of them is pathological then $X \oplus Y$ is also pathological.

Corollary 8. For any uncountable cardinal κ, there is a pathological pre-Hilbert space X of dimension and density κ.

Proof. Let X_0 be Halmos' pre-Hilbert space with density $\aleph_1$. By Corollary 7, $X = X_0 \oplus \ell_2(\kappa)$ will do. $\square$

Another construction of pathological pre-Hilbert spaces

Theorem 9. Assume that $\mathrm{ADS}^-(\kappa)$ holds for a regular cardinal $\kappa > \omega_1$. Then there is a pathological linear subspace X of $\ell_2(\kappa)$ dense in $\ell_2(\kappa)$ s.t. $X \downarrow \beta$ is non-pathological for all $\beta < \kappa$. Furthermore for any regular $\lambda < \kappa$, $\{S \in [\kappa]^\lambda : X \downarrow S$ is non-pathological$\}$ contains a club subset of $[\kappa]^\lambda$.

Remark 10. The theorem above implies that the Fodor-type Reflection Principle follows from the global reflection of pathology of pre-Hilbert spaces down to subspaces of density $< \aleph_2$.

Sketch of the proof of Theorem 9: Let $\langle A_\alpha : \alpha \in E\rangle$ be an $\mathrm{ADS}^-(\kappa)$-sequence on a stationary $E \subseteq E^\omega_\kappa$.

- Let $\langle \mathbb{u}_\xi : \xi < \kappa\rangle$ be a sequence of elements of $\ell_2(\kappa)$ s.t.
  (1) $\mathbb{u}_\xi = \mathbb{e}^\kappa_\xi$ for all $\xi \in \kappa \setminus E$,
  (2) $\mathrm{supp}(\mathbb{u}_\xi) = A_\xi \cup \{\xi\}$ for all $\xi \in E$.
- Let $U = \{\mathbb{u}_\xi : \xi < \kappa\}$ and $X = [U]_{\ell_2(\kappa)}$.
- This X is as desired. $\square$

Singular Compactness

- The following theorem can be proved analogously to the proof of the Shelah Singular Compactness Theorem given in [Hodges, 1981]:

Theorem 11. Suppose that λ is a singular cardinal and X is a pre-Hilbert space which is a dense sub-inner-product-space of $\ell_2(\lambda)$. If X is pathological then there is a cardinal $\lambda' < \lambda$ s.t.

(1) $\{u \in [\lambda]^{\kappa^+} : X \downarrow u$ is a pathological pre-Hilbert space$\}$

is stationary in $[\lambda]^{\kappa^+}$ for all $\lambda' \leq \kappa < \lambda$.

Fodor-tpye Reflection Principle

Theorem 12. TFAE over ZFC:

(a) Fodor-type Reflection Principle (FRP);

(b) For any regular $\kappa > \omega_1$ and any linear subspace X of $\ell_2(\kappa)$ dense in $\ell_2(\kappa)$, if X is pathological then

$$S_X = \{\alpha < \kappa \,:\, X \downarrow \alpha \text{ is pathological}\} \tag{1}$$

is stationary in κ;

(c) For any regular $\kappa > \omega_1$ and any dense sub-inner-product-space X of $\ell_2(\kappa)$, if X is pathological then

$$S_X^{\aleph_1} = \{U \in [\kappa]^{\aleph_1} \,:\, X \downarrow U \text{ is pathological}\} \tag{2}$$

is stationary in $[\kappa]^{\aleph_1}$.

Proof. "(a) $\Rightarrow$ (b), (c)": By induction on $d(X)$. Use Theorem 11 for singular cardinal steps.

- "$\neg$(a) $\Rightarrow \neg$(b) $\wedge$ $\neg$(c)": By Theorem 9 and Theorem 11a (on the extra slide with the definition of FRP). □

FRP is a "mathematical reflection principle"

- The FRP is known to be equivalent to each of the following "mathematical" assertions:

(A) For every locally countably compact topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable.

(B) Any uncountable graph G has countable coloring number if all induced subgraphs of G of cardinality $\aleph_1$ have countable coloring number.

(C) For every Boolean algebra B, if there are club many subalgebras of B of cardinality $\aleph_1$ which are openly generated then B itself is also openly generated.

Further reflections

- There are many open problems around the minimal cardinal numbers κ_A, κ_B, κ_C with the following properties:

  (A') For every ~~locally countably compact~~ first countable topological space X, if all subspaces of X of cardinality $< \kappa_A$ are metrizable, then X itself is also metrizable.

  (B') Any uncountable graph G has countable ~~coloring~~ chromatic number if all induced subgraphs of G of cardinality $< \kappa_B$ have countable ~~coloring~~ chromatic number.

  (C') For every Boolean algebra B, if there are club many subalgebras of B of cardinality $< \kappa_C$ which are ~~openly generated~~ free then B itself is also ~~openly generated~~ free.

- ⓐ consis($\kappa_A = \aleph_2$) is still open and is known as Hamburger's problem. ⓑ $\kappa_B > \beth_\omega$ by a theorem of Erdös and Hajnal. ⓒ κ_C is possibly above a large cardinal (Saharon Shelah should already know much about it).

In a pre-Hilbert space
a maximal orthonormal system
need not to be an independent basis.

Gràcies per la seva atenció.

[1] Sakaé Fuchino, Pre-Hilbert spaces without orthonormal bases, submitted.
https://arxiv.org/pdf/1606.03869v2

Ramon LLull (1232–1315)

Coloring number of a graph

- A graph $E = \langle E, K \rangle$ has **coloring number** $\leq \kappa \in \mathrm{Card}$ if there is a well-ordering $\sqsubseteq$ on E s.t. for all $p \in E$ the set

$$\{q \in E \ : \ q \sqsubseteq p \text{ and } q \, K \, p\}$$

has cardinality $< \kappa$.

- The coloring number $col(E)$ of a graph E is the minimal cardinal among such κ as above.

Back

Notation: $\ell_2(S)$ and its standard unit vectors

- For an infinite set S, let

  (1) $\boldsymbol{\ell_2(S)} = \{\mathbb{u} \in {}^S K \ : \ \sum_{x\in S}(\mathbb{u}(x))^2 < \infty\}$,

  where $\sum_{x\in S}(\mathbb{u}(x))^2$ is defined as $\sup\{\sum_{x\in A}(\mathbb{u}(x))^2 \ : \ A \in [S]^{<\aleph_0}\}$.

- $\ell_2(S)$ is a/the Hilbert space of density $|\,S\,|$ endowed with a natural structure of inner product space with coordinatewise addition and scalar multiplication, the zero element $\mathbb{0}_{\ell_2(S)}$ with $\mathbb{0}_{\ell_2(S)}(s) = 0$ for all $s \in S$, as well as the inner product defined by

  (2) $(\mathbb{u}, \mathbb{v}) = \sum_{x\in S} \mathbb{u}(x)\overline{\mathbb{v}(x)}$ for $\mathbb{u}$, $\mathbb{v} \in \ell_2(S)$.

- For $x \in S$, let $\boldsymbol{\mathbb{e}^S_x} \in \ell_2(S)$ be the standard unit vector at x defined by

  (3) $\mathbb{e}^S_x(y) = \delta_{x,y}$ for $y \in S$.

- $\{\mathbb{e}^S_x \ : \ x \in S\}$ is an orthonormal basis of $\ell_2(S)$.

Back

Notation: Support of elements of $\ell_2(S)$ and direct sum of Hilbert spaces

- For $\mathbb{a} \in \ell_2(S)$, the **support** of $\mathbb{a}$ is defined by

  (1) $\mathrm{supp}(\mathbb{a}) = \{x \in S : \mathbb{a}(x) \neq 0\}$ $(= \{x \in S : (\mathbb{a}, \mathbb{e}_x^S) \neq 0\})$.

- ▷ By the definition of $\ell_2(S)$, $\mathrm{supp}(\mathbb{a})$ is a countable subset of S for all $\mathbb{a} \in \ell_2(S)$.

- For any two pre-Hilbert spaces X, Y, the **orthogonal direct sum of X and Y** is the direct sum $X \oplus Y = \{\langle \mathbb{x}, \mathbb{y}\rangle : \mathbb{x} \in X, \mathbb{y} \in Y\}$ of X and Y as linear spaces together with the inner product defined by $(\langle \mathbb{x}_0, \mathbb{y}_0\rangle, \langle \mathbb{x}_1, \mathbb{y}_1\rangle) = (\mathbb{x}_0, \mathbb{x}_1) + (\mathbb{y}_0, \mathbb{y}_1)$ for $\mathbb{x}_0$, $\mathbb{x}_1 \in X$ and $\mathbb{y}_0$, $\mathbb{y}_1 \in Y$.

- A sub-inner-product-space X_0 of a pre-Hilbert space X is an **orthogonal direct summand of X** if there is a sub-inner-product-space X_1 of X s.t. the mapping $\varphi : X_0 \oplus X_1 \to X$; $\langle \mathbb{x}_0, \mathbb{x}_1\rangle \mapsto \mathbb{x}_0 + \mathbb{x}_1$ is an isomorphism of pre-Hilbert spaces. If this holds, we usually identify $X_0 \oplus X_1$ with X by φ as above.

Back

Notation: $X \downarrow S$, $\overline{\bigoplus}_{i\in I} X_i$ etc.

- For $X \subseteq \ell_2(S)$ and $S' \subseteq S$, let $X \downarrow S' = \{\mathbb{u} \in X \,:\, \mathrm{supp}(\mathbb{u}) \subseteq S'\}$.
- For $\mathbb{u} \in \ell_2(S)$, let $\mathbb{u} \downarrow S' \in \ell_2(S)$ be defined by, for $x \in S$,

$$(\mathbb{u} \downarrow S')(x) = \begin{cases} \mathbb{u}(x) & \text{if } x \in S' \\ 0 & \text{otherwise.} \end{cases}$$

▷ Note that $X \downarrow S'$ is not necessarily equal to $\{\mathbb{u} \downarrow S' \,:\, \mathbb{u} \in X\}$

- A sub-inner-product-space X_0 of a pre-Hilbert space X is an **orthogonal direct summand of X** if there is a sub-inner-product-space X_1 of X s.t. the mapping $\varphi : X_0 \oplus X_1 \to X;\ \langle \mathbb{x}_0, \mathbb{x}_1 \rangle \mapsto \mathbb{x}_0 + \mathbb{x}_1$ is an isomorphism of pre-Hilbert spaces. If this holds, we usually identify $X_0 \oplus X_1$ with X by φ as above.

- For pairwise orthogonal linear spaces X_i, $i \in I$ of X, we denote with $\overline{\bigoplus}^X_{i\in I} X_i$ the maximal linear subspace X' of X s.t. X' contains $\bigoplus_{i\in I} X_i$ as a dense subset of X'. Thus, we have $X = \overline{\bigoplus}^X_{i\in I} X_i$ if $\bigoplus_{i\in I} X_i$ is dense in X. If it is clear in which X we are working we drop the superscript X and simply write $\overline{\bigoplus}_{i\in I} X_i$.

Dimension of a pre-Hilbert space

- Let X be a pre-Hilbert space. By Bessel's inequality, all maximal orthonormal system of X have the same cardinality.

  ▷ This cardinality is called the **dimension** of X and denoted by $\dim(X)$.

- $\dim(X) \leq d(X)$.

- Note that, if $\dim(X) < d(X)$, then X cannot have any orthonormal basis.

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The proof of Proposition 4.

Proposition 4. (D. Buhagiara, E. Chetcutib and H. Weber 2008) For any pre-Hilbert space X, we have $d(X) \leq |X| \leq (\dim(X))^{\aleph_0}$.

Proof. Let X be a pre-Hilbert space with $d(X) = \lambda \leq \kappa = \dim(X)$. Wlog we may assume that X is a dense subspace of $\ell_2(\lambda)$ and $\kappa \geq \aleph = 0$.

- Let $\mathcal{B} = \langle \mathbb{b}_\xi : \xi < \kappa \rangle$ be a maximal orthonormal system in X and $D = \bigcup\{\mathrm{supp}(\mathbb{b}_\xi) : \xi < \kappa\}$. By the assumption we have $|D| = \kappa$.
- For any distinct $\mathbb{a}_0, \mathbb{a}_1 \in X$ we have $\mathbb{a}_0 \restriction D \neq \mathbb{a}_1 \restriction D$.
- Then $\varphi : \ell_2(D) \to X$ defined by

$$\varphi(\mathbb{c}) = \begin{cases} \text{the unique } \mathbb{a} \in X \text{ s.t. } \mathbb{c} = \mathbb{a} \restriction D; & \text{if there is such } \mathbb{a} \in X, \\ \mathbb{0}; & \text{otherwise} \end{cases}$$

  is well defined and surjective. Thus
- $d(X) \leq |X| \leq |\ell_2(D)| = (\dim(X))^{\aleph_0}$. $\square$

Back

$\mathsf{ADS}^-(\kappa)$ and $\mathsf{ADS}^-(\kappa)$-sequence

- For a regular cardinal κ, $\mathsf{ADS}^-(\kappa)$ is the assertion that there is a stationary set $E \subseteq E^\omega_\kappa$ and a sequence $\langle A_\alpha : \alpha \in E\rangle$ s.t.
  (1) $A_\alpha \subseteq \alpha$ and $\mathrm{ot}(A_\alpha) = \omega$ for all $\alpha \in E$;
  (2) for any $\beta < \kappa$, there is a mapping $f : E \cap \beta \to \beta$ s.t. $f(\alpha) < \sup(A_\alpha)$ for all $\alpha \in E \cap \beta$ and $A_\alpha \setminus f(\alpha)$, $\alpha \in E \cap \beta$ are pairwise disjoint.

- We shall call $\langle A_\alpha : \alpha \in E\rangle$ as above an $\mathsf{ADS}^-(\kappa)$-**sequence**.

- Note that it follows from (1) and (2) that A_α, $\alpha \in E$ are pairwise almost disjoint.

Back

FRP

(FRP) For any regular $\kappa > \omega_1$, any stationary $S \subseteq E^{\omega}_{\kappa}$ and any mapping $g : S \to [\kappa]^{\aleph_0}$, there is $\alpha^* \in E^{\omega_1}_{\kappa}$ s.t.

(*) α^* is closed w.r.t. g (that is, $g(\alpha) \subseteq \alpha^*$ for all $\alpha \in S \cap \alpha^*$) and, for any $I \in [\alpha^*]^{\aleph_1}$ closed w.r.t. g, closed in α^* w.r.t. the order topology and with $\sup(I) = \alpha^*$, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in S$ and $g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$

Theorem 11a (S.F., H.Sakai and L.Soukup) TFAE over ZFC:

(a) FRP;

(b) $\mathsf{ADS}^-(\kappa)$ does not hold for all regular uncountable $\kappa > \omega_1$.

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