A Geometric Condition of Extender Based Prikry Forcing

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Background

Prikry forcing was introduced in the year 1970. It enables us to construct a model which convert a inaccessible cardinal to a singular cardinal. Let κ be a measurable cardinal and U a normal ultrafilter over κ.

Definition (Prikry forcing)

Let P be the set of all pairs $\langle p, A \rangle$ such that p is a finite subset of κ, $A \in U$ and $min(A) > max(p)$. Suppose that $\langle p, A \rangle$, $\langle q, B \rangle \in P$, $\langle p, A \rangle \leq \langle q, B \rangle$ iff p is an end extension of q, $A \subseteq B$ and $p \setminus q \subseteq B$.

Background

After that, as the developing of the techniques, Prikry type forcings become one of the most powerful tools to get the consistency results.

- Gitik showed that all the limit ordinals may of cofinality ω with the absence of AC by assuming the existence of a proper class of strongly compact cardinals is compatible with ZFC.
- Magidor showed that the negation of SCH is compatible with ZFC from the consistency of supercompactness.

What is geometric condition

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Theorem (Prikry condition)

For every given $p \in P$ and a formula Φ with all its parameters in κ, there is a $q \in P$, $q \leq^* p$ (p and q shares the same stem with p) and $q \parallel \Phi$.

In fact, it is possible to get something more than that.

What is geometric condition

In the language of forcing, even though the genericity is a concept with respect to all dense subsets, the non-empty intersection of the generic and several certain dense sets is suffecient to verify the validity of required properties in the generic model.
In fact, for most forcings, it is very difficult to tell all the properties that a generic object can have, since it is not easy to figure out the pattern of all the dense subsets.

What is geometric condition

However, for most Prikry type forcings, it is possible to get a characterization of the generic from the measure in the ground. In another word, the measure in the ground can give us full information of all the dense sets. Such property is called geometric condition.

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Theorem (Mathias condition for Prikry forcing)

For every $G \subseteq P$, let g be the set $\bigcup\{p \mid \exists B \langle p, B\rangle \in G\}$, then G is a generic set iff for all $A \in U$, $g \setminus A$ is finite.

Applications of geometric conditions

Prikry-type forcing	Application of its geometric condition
Prikry forcing with respect to $i_{0,\omega}(U)$ in Ult^{ω}	$\bigcap_{n<\omega} Ult^n = Ult^{\omega}[\langle i_{0,n}(\kappa) \mid n < \omega \rangle]$ (Bukovský [1] [2])
Force with a uniform ultrafilter over ω_1	Analyze the properties of $\mathbb{P}^{\clubsuit NS}_{max}$ (Woodin [10])
Diagonal Prikry forcings with and without guiding generic	Analyse the combinatoric properties including weakly square (Sinapova [8])
Extender-based Prikry forcing with a single Extender	$Con(I_0)$ implies $Con(I_1 + \neg SCH$ at $\kappa)$ (Dimonte - Wu[3] and Shi - Trang[9])

Table: several applications of geometric condition

Failure of SCH

- $Con(ZFC + \exists\kappa(\kappa \text{ is supercompact})) \to Con(\neg SCH_\kappa)$(Magidor)
- $Con(ZFC + \exists\kappa(o(\kappa) = \kappa^{++})) \to Con(\neg SCH_\kappa)$(Gitik)
- $Con(ZFC + \exists\kappa(\kappa \text{ is a countable limit of strong cardinals}))$
  $\to Con(\neg SCH_\kappa)$(Gitik)
- $Con(ZFC + \exists\kappa(\kappa \text{ is a countable limit of "short" strong cardinals}))$
  $\to Con(\neg SCH_\kappa)$(Gitik)

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Furthermore Gitik and Mitchell showed that if we want to blow up the power set of a strong limit cardinal κ to be λ, we need either the consistency of $o(\kappa) = \lambda$ or the consistency of existence of a cofinal sequece to κ with increasing Mitchell orders.

One element Prikry forcing

Let κ be the limit of an increasing sequence $\langle \kappa_n | n < \omega \rangle$ of measurable cardinals and each κ_n is equipped with a normal measure U_n. Let $Q_n = U_n \cup \kappa_n$.

One-element forcing

The partial order $\mathbb{P}$ satisfies that every member $p \in \mathbb{P}$ is an element of $\langle Q_n | n < \omega \rangle$ such that there is a natural number n so that $((\forall m < n(p(m) \in \kappa_m)) \wedge (\forall m > n(p(m) \in U_n)))$. And $p \leq q$ iff for all natural number n

- $p(n) = q(n)$ if $q(n)$ is an ordinal, or
- $p(n) \in q(n)$ if $q(n) \in U_n$ and $p(n)$ is an ordinal,
- $p(n) \subseteq q(n)$ otherwise.

One element Prikry forcing

The behavior of the generics over this partial order is very clear:

Prikry condition for one element Prikry forcing

Let $p \in \mathbb{P}$ and D is an open dense subset of $\mathbb{P}$ below p. Then there is a p^* such that $p^* \leq^* p$ (i.e. $l(p^*) = l(p)$) and $n^* < \omega$ so that for every $\langle \nu_0, \dots, \nu_{n^*-1} \rangle \in \prod_{l(p)<m<l(p)+n^*-1} A_m(p^*)$, $p^* \frown \langle \nu_0, \dots, \nu_{n^*-1} \rangle$ is in D.

$p^* \frown \langle \nu_0, \dots, \nu_{n^*-1} \rangle$ is defined to be the condition which assigns $q(i)$ to ν_i for every i, $l(p) < i < l(p) + n^* - 1$ and to $p^*(i)$ otherwise.

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Based on this proposition we can obtain:

Geometric condition for one element Prikry forcing

Suppose that $\langle f(n) \mid n < \omega \rangle$ is a countable cofinal sequence of κ, f is generated by the generic object iff for every sequence $\langle A_n \mid n < \omega \rangle$ such that $\forall n < \omega (A_n \in U_n)$, $f(m)$ is in A_m for all large number m.

Definitions of the extender based forcing

Assumptions

Suppose κ is the limit of the sequence $\langle \kappa_n \mid n < \omega \rangle$ such that each κ_n is $(\kappa_n, \lambda + 1)-$strong for a regular cardinal $\lambda \geq \kappa^+$.

1. Let E_n be a $(\kappa_n, \lambda + 1)-$extender.
2. For $\alpha < \lambda$, $\mathbf{U}_{n,\alpha}$ is the ultrafilter generated by the seed $\{\alpha\}$ via j_{E_n}.
3. For $\alpha,\beta < \lambda$ define $\alpha \leq_{E_n} \beta$ if and only if $\alpha \leq \beta$ and for some $f \in {}^{<\kappa_n}\kappa_n$, $j_{E_n}(f)(\beta) = \alpha$. for such $\alpha \leq_{E_n} \beta$. Therefore we can fix a set $\{\pi_{\beta\alpha} \mid \beta < \lambda,\ \alpha < \lambda,\ j_{E_n}(\pi_{\beta\alpha})(\beta) = \alpha\}$.
4. $\pi_{\beta\alpha}$ will induce a Rudin-Keisler projection from $\mathbf{U}_{n,\beta}$ to $\mathbf{U}_{n,\alpha}$.

Definitions of the extender based forcing

Partial order at stage n

Let $Q_{n1} = \{f \mid f$ is a partial function from λ to κ_n of cardinality at most $\kappa\}$.
The order $\leq_1$ on Q_{n1} is inclusion.
Let Q_{n0} be the set of triples $\langle a, A, f\rangle$ so that:

1. $f \in Q_{n1}$,
2. $a \subseteq \lambda$ with $|a| < \kappa_n$, $a \cap dom(f) = \emptyset$, and a has a $\leq_{E_n} -$ maximal element,
3. $A \in \mathbf{U}_{n,max(a)}$,
4. $\forall(\alpha, \beta, \gamma \in a \wedge \alpha >_{E_n} \beta >_{E_n} \gamma)\forall\rho(\rho \in \pi''_{mc(a)\alpha}A)$ $(\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$.
5. $\forall\alpha \in a\forall\beta \in a\forall\nu \in A((\alpha > \beta) \rightarrow (\pi_{max(a)\alpha}(\nu) > \pi_{max(a)\beta}(\nu)))$.

For $\langle a, A, f\rangle\langle b, B, g\rangle \in Q_{n0}$ $\langle a, A, f\rangle \leq_0 \langle b, B, g\rangle$ if and only if $f \supseteq g$, $a \supseteq b$ and $\pi''_{max(a),max(b)}A \subseteq B$.

Definitions of the extender based forcing

Partial order at stage n

Let $Q_n = Q_{n0} \cup Q_{n1}$. Then $p \leq q$ if and only if

- $p \leq_0 q$, or
- $p \leq_1 q$, or
- p is in Q_{n1} and $q = \langle a, A, f \rangle$ is in Q_{n0} such that $p \supseteq f$, $dom(p) \supseteq a$, $p(max(a)) \in A$ and $\forall \beta \in a(p(\beta) = \pi_{max(a)\beta}(p(max(a))))$.

Definitions of the extender based forcing

The partial order $\mathbb{P}$

The set $\mathbb{P}$ consists of sequences $p = \langle p_n \mid n < \omega \rangle$ such that for every $n < \omega$, $p_n \in Q_n$ and there is a $l(p) < \omega$ such that

1. for every $n < l(p)$, $p_n \in Q_{n1}$,
2. and for every $n > l(p) - 1$, $p_n = \langle a_n, A_n, f_n \rangle \in Q_{n0}$ and $a_n \subseteq a_{n+1}$.

Let $p = \langle p_n \mid n < \omega \rangle$ and $p = \langle q_n \mid n < \omega \rangle \in \mathbb{P}$. We set $p \leq q$ iff for every $n < \omega$, $p_n \leq_{Q_n} q_n$.

Theorem (Prikry condition see [4] Lemma 2.8)

Let $p \in \mathbb{P}$ and D is an open dense subset of $\mathbb{P}$ below p. Then there are $p^* \leq^* p$ and $n^* < \omega$ so that for every $\langle \nu_0, \ldots, \nu_{n^*-1} \rangle \in \prod_{l(p)<m<l(p)+n^*-1} A_m(p^*)$, $p^* \frown \langle \nu_0, \ldots, \nu_{n^*-1} \rangle \in D$.

$p^* \frown \langle \nu_0, \ldots, \nu_{n^*-1} \rangle$ is defined to be the condition extends p^* by assigning $q(i)$ to be $p^*(i)$ for every i, $i < l(p^*) \vee i \geq l(p^*) + \nu_{n^*-1}$ and assigning $q(i)$ to be the union of the $f_i(p^*)$ and the function with domain $a_i(p^*)$ which takes the value computed by the Rudin-Keisler projection and $(max(a_i(p^*)), \nu_i)$ for every i, $l(p) < i < l(p) + n^* - 1$.

The generic object of extender based forcing

Definition of the baby forcing

Let $\mathbb{P}_0$ be a partial order consists of the conditions $\langle f_i \mid i < \omega \rangle$ such that for every i,

- f_i is a function from λ to 2,
- $|f_i^{-1}{}''\{0\}| < \kappa_i$ and
- $|f_i^{-1}{}''\{1\}| \leq \kappa$.

The relation $\leq$ be the including relation for all sufficient large natural coordinates.

The generic object of extender based forcing

Geometric condition for extender based forcing

g is a generic function over $\mathbb{P}$ if and only if there is a G_0 such that the following hold:

(i) g and G_0 are mutually generic.

(ii) (coherency) Let g_i be $g \restriction i$, if $p = \langle f_i \mid i < \omega \rangle \in G_0$ (w.o.l.g. by genericity we can assume for all i $f_i^{-1"}\{0\}$ has a $\leq_{E_i}$ maximal element) then there is a natural number m such that $\forall i \geq m \forall x \in f_i^{-1"}\{0\}(g_i(x) = \pi_{max(f_i^{-1"}\{0\}),x} g_i(max(f_i^{-1"}\{0\})))$ and we say m is a witness of coherency between p and g.

(iii) (one-element genericity) Suppose $p = \langle f_i \mid i < \omega \rangle$ $p \in G_0$ then for every sequence $\langle A_n \mid n < \omega \rangle$ such that $A_i \in \mathbf{U}_{i, max(f_i^{-1"}\{0\})}$ there is a natural number n such that $\forall j > m(g_j(max(f_i^{-1"}\{0\})) \in A_j)$, we call m a witness of the one-element genericity between p and g.

Note that the geometric condition actually tells us the extender based Prikry forcing is a combination of a one element forcing and a almost Cohen forcing.

The generic object of extender based forcing

The sketch of the proof:

- Let G be the set $\{\langle f_i \mid i < l\rangle^\frown\langle (h_i^{-1``}\{0\}, A_i, g_i \restriction h_i^{-1``}\{1\}) \mid i \geq l\rangle \mid f_i \subseteq g_i, |f_i| < \kappa, \langle\ h_i \mid i < \omega\rangle \in G_0, A_i \in \mathbf{U}_{i,mc(a_i)}$ and l witnesses (ii) and (iii) $\}$

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- Goal: G is a generic over P. Fix a dense open set D, we have to prove $D \cap G \neq \emptyset$.
- (i) helps us to transfer the generic from P_0 to P.
- Problem: Since the information of the measure is isolated from the almost Cohen part, it is impossible to tell how to extend the stem of a condition. That's to say, if we go into P_0, the information of the length of the condition in P is missed.
- We can enumerate all the functions at the initial stages and combine all of the answers from (i) after putting in the conditions which contain them as their stems to get a universal condition. The universal condition is responsible for recovering the information of length.

Gap two short extender forcing

From now on we assume that κ is the limit of the sequence $\langle \kappa_n | n < \omega \rangle$ such that each κ_n is $(\kappa_n, \lambda_n)-$strong where $\lambda_n = \kappa_n^{+n+2}$.

1. Let E_n be a $(\kappa_n, \lambda_n)-$extender,
2. For $\alpha < \lambda$, $\mathbf{U}_{n,\alpha}$ is the ultrafilter generated by the seed $\{\alpha\}$ via j_{E_n},
3. For $\alpha,\beta < \lambda$ define $\alpha \leq_{E_n} \beta$ if and only if $\alpha \leq \beta$ and for some $f \in {}^{<\kappa_n}\kappa_n$, $j_{E_n}(f)(\beta) = \alpha$. for such $\alpha \leq_{E_n} \beta$. Therefore we can fix a set $\{\pi_{\beta\alpha} \mid \beta < \lambda,\ \alpha < \lambda,\ j_{E_n}(\pi_{\beta\alpha})(\beta) = \alpha\}$.
4. $\pi_{\beta\alpha}$ will induce a Rudin-Keisler projection from $\mathbf{U}_{n,\beta}$ to $\mathbf{U}_{n,\alpha}$.

We can attempt to construct a forcing by analogizing the former one. But the obstacle is the chain condition that responsible for preserving κ^{++}.

Gap two short extender forcing

To fix the chain condition, the definition of the partial order is based on some model theoretic notions.

Gap two short extender forcing

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Model theoretic notions

Fix $n < \omega$. For every $k < n+1$ we consider a language $\mathfrak{L}_{n,k}$ containing two relation symbols, a function symbol, a constant c_α for every $\alpha < \kappa_n^{+k}$ and constants c_{λ_n},c. The structure $\mathfrak{a}_{n,k}$ is defined to be

$\langle H(\chi^{+k}), \in, E_n, \text{the enumeration of } [\lambda_n]^{<\lambda_n}, \lambda_n, \chi, 0, 1, ..., \alpha... \mid \alpha < \kappa_n^{+k} \rangle$

where χ is a regular cardinal large enough. For an ordinal $\eta < \chi$ we denote by $tp_{n,k}(\eta)$ the $\mathfrak{L}_{n,k}-$type realized by η in $\mathfrak{a}_{n,k}$. Similarly for every $\delta < \chi$ we can define $tp_{n,k}(\delta, \eta)$.

Gap two short extender forcing

Model theoretic notions

let $k < n+1$ and $\beta < \lambda_n$. β is called *k-good* iff

- for every $\gamma < \beta$, $tp_{n,k}(\gamma, \beta)$ is realized unboundedly many times below λ_n,
- for every $a \subseteq \beta$ of $|a| < \kappa_n$ then there is $\alpha < \beta$ corresponding to a in the enumeration of $[\lambda_n]^{<\lambda_n}$.

β is *good* if it is *k-good* for some $k < n+1$.

- The concept of goodness enables us to define an equivalent relation on the partial order.
- In each equivalent class Rudin-Keisler projection is preserved.

Gap two short extender forcing

Partial order at stage n

Let $Q_{n1} = \{f \mid f$ is a partial function from λ to κ_n of cardinality at most $\kappa\}$. We order Q_{n1} by inclusion, which here is denoted by $\leq_1$.
Let Q_{n0} be the set of triples $p = \langle a, A, f\rangle$ so that:

1. $f \in Q_{n1}$,
2. a is a order-preserving function from κ^{++} to λ_n with $|a| < \kappa_n$, $a \cap dom(f) = \emptyset$, and $dom(a)$ has a maximal element,
3. $A \in \mathbf{U}_{n,max(ran(a))}$ and $(\forall(\alpha, \beta, \gamma \in a \wedge \alpha >_{E_n} \beta >_{E_n} \gamma)$
$\forall\rho(\rho \in \pi''_{max(a)\alpha}A)(\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$,
4. $\forall\alpha \in ran(a)\forall\beta \in ran(a)\forall\nu \in A((\alpha > \beta) \rightarrow$
$(\pi_{max(ran(a))\alpha}(\nu) > \pi_{max(ran(a))\beta}(\nu)))$.

For $\langle a, A, f\rangle\langle b, B, g\rangle \in Q_{n0}$ $\langle a, A, f\rangle \leq_0 \langle b, B, g\rangle$ if and only if $f \supseteq g$, $a \supseteq b$ and $\pi''_{max(ran(a)),max(ran(b))}A \subseteq B$.

Gap two short extender forcing

The underlying set of partial order $\mathbb{P}$

The set $\mathbb{P}$ consists of sequences $p = \langle p_n \mid n < \omega \rangle$ such that for every $n < \omega$, $p_n \in Q_n$ and there is a $l(p) < \omega$ such that

1. for every $n < l(p)$, $p_n \in Q_{n1}$,
2. and for every $n > l(p) - 1$, $p_n = \langle a_n^p, A_n^p, f_n^p \rangle \in Q_{n0}$ and $dom(a_n) \subseteq dom(a_{n+1})$,
3. For every n, $l(p) \leq n < \omega$ and $\beta \in dom(a(n))$ there exist a non-decreasing sequence $\langle k_m \mid n \leq m < \omega \rangle$ with an infinite limit so that for every $m \geq n$, $a_m(\beta)$ is k_m-good.

Gap two short extender forcing

Definition (Direct extension)

Let $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle \in \mathbb{P}$. $p \leq^* q$ iff

1. $l(p) = l(q)$,
2. for every $l \leq n < \omega$ $a^p_n \subseteq a^q_n$, $f^p_n \subseteq f^q_n$,
3. $dom(a^p_n) \setminus dom(a^q_n) \subseteq dom(f^p_n) \setminus dom(f^q_n)$,
4. $A^p_n \subseteq \pi^{-1''}_{max(ran(a^p_n)),max(ran(a^q_n))} A^q_n$.

Gap two short extender forcing

Definition (extension)

Let $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle \in \mathbb{P}$. $p \leq q$ iff

1. for every $m < l(q)$ q_n is a subset of p_n,
2. for every m, $l(q) \leq m < l(p)$ $p(m)$ is the interpretation of $q(m)$ with an element in A^q_m,
3. for every $l(p) \leq n < \omega$ $a^p_n \subseteq a^q_n$, $f^p_n \subseteq f^q_n$,
4. $dom(a^p_n) \setminus dom(a^q_n) \subseteq dom(f^p_n) \setminus dom(f^q_n)$,
5. $A^p_n \subseteq \pi^{-1"}_{max(ran(a^p_n)),max(ran(a^q_n))} A^q_n$,
6. there exist a non-decreasing sequence $\langle k_n | l(p) \leq n < \omega \rangle$ $tp_{n,k_n}(ran(a^p_n \restriction dom(a^q_n)) = tp_{n,k_n}(ran(a^q_n))$.

Gap two short extender forcing

Suppose $A = \{\langle a, \langle b, c\rangle\rangle \mid a, b, c \in V\}$, for simplicity, I denote $A^{-1"}\{c\}$ to be the set $\{\langle a, b\rangle \mid \langle a, \langle b, c\rangle\rangle \in A\}$ and $A^{-2"}\{c\}$ to be the set $\{a \mid \langle a, \langle b, c\rangle\rangle \in A\}$.

The baby partial order $\mathbb{P}_0$

We say p is a condition in $\mathbb{P}_0$ iff p is a sequence $\langle f_i \mid i < \omega\rangle$ such that for every i,

- f_i is a function from λ to $\kappa_i^{+i+2} \times 2$,
- $|f_i^{-2"}\{0\}| < \kappa_i$ and $|f_i^{-2"}\{1\}| \leq \kappa$,
- $f_i^{-1"}\{0\}$ is an order preserving function such that there is a non-decreasing sequence of natural numbers with infinite limit which witnesses the goodness of the range of $f_i^{-1"}\{0\}$.

The relation $\leq$ be the relation induced by $\leq$ on $\mathbb{P}$.

The Prikry condition is due to Merimovich see [7]

Theorem

Let $p \in \mathbb{P}$ and D is an open dense subset of $\mathbb{P}$ below p. Then there are $p^* \leq^* p$ and $n^* < \omega$ such that for every $\langle \nu_0, \ldots, \nu_{n^*-1} \rangle \in \prod_{l(p)<m<l(p)+n^*-1} A_m(p^*)$, $p^* \frown \langle \nu_0, \ldots, \nu_{n^*-1} \rangle \in D$.

Gap two short extender forcing

Geometric condition for gap two extender forcing

g is a generic function over $\mathbb{P}$ if and only if there is a G_0 such that the following hold:

(i) g and G_0 are mutually generic.

(ii) (coherency) Let g_i be $g \restriction i$,if $p = \langle f_i \mid i < \omega \rangle \in G_0$ (w.o.l.g. by genericity we can assume for all i $ran(f_i^{-1"})\{0\}$ has a $\leq_{E_i}$ maximal element) then there is a natural number m such that
$\forall i \geq m \forall x \in f_i^{-2"}\{0\}(g_i(x) = \pi_{max(ran(f_i^{-1"}\{0\})),x} g_i(max(ran(f_i^{-1"}\{0\}))))$ and we say m is a witness of coherency between p and g.

(iii) (one-element genericity)Suppose $p = \langle f_i \mid i < \omega \rangle$ $p \in G_0$ then for every sequence $\langle A_n \mid n < \omega \rangle$ such that $A_i \in \mathbf{U}_{i,max(ran(f_i^{-1"}\{0\}))}$ there is a natural number n such that $\forall j > m(g_j(max(ran(f_i^{-1"}\{0\}))) \in A_j)$, we call m a witness of the one-element genericity between p and g.

(iv) for a given sequence $\langle p_\alpha \mid \alpha < \beta < \kappa_n \rangle$ if it satisfies $\forall \alpha (p_\alpha \in G_0 \wedge n$ is a witness of coherency between p_α and g) then there is a $p \in G_0 \wedge$ n is a witness of coherency between p and g such that $\forall (i > n) \forall \alpha (dom^2(p_\alpha(i)) \subseteq dom^2(p(i)))$.

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Thank you