Tutorial: Ramsey theory in Forcing

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Outline of Tutorial

Day 1

1. Introduction to topological Ramsey spaces
2. Classes of new topological Ramsey spaces which are dense in σ-closed forcings yielding ultrafilters with complete combinatorics

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Day 3

1. Topological Ramsey spaces of strong trees
2. Applications to finding finite Ramsey degrees for universal relational structures, including the universal triangle-free graph

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Graphics by Timothy Trujillo.

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5. New topological Ramsey spaces dense inside σ-closed forcings which add ultrafilters satisfying weak partition properties
6. A new Ramsey theorem motivated by this study

Ramsey's Theorem and Higher Dimensional Versions

Thm. (Ramsey) Given $n, l \geq 1$ and a coloring $c : [\omega]^n \to l$, there is an infinite set $M \subseteq \omega$ such that c is monochromatic on $[M]^n$.

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Thm. (Nash-Williams) Every Nash-Williams set $\mathcal{F} \subseteq [\omega]^{<\omega}$ is Ramsey: Given a coloring c on a front $\mathcal{F}$ into 2 colors, there is an $M \in [\omega]^\omega$ such that c is monochromatic on $\mathcal{F}|M := \{a \in \mathcal{F} : a \subseteq M\}$.

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The optimal extension of Ramsey's Theorem to infinite dimensions is Ellentuck's Theorem.

This theorem uses a topology on the Baire space which refines the metric topology.

Ellentuck Space $([\omega]^\omega, \subseteq, r)$

Basis for topology: $[a, X] = \{Y \in [\omega]^\omega : a \sqsubset Y \subseteq X\}$.

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Thm. (Ellentuck) Every $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire is Ramsey, and every meager set is Ramsey null.

Connection with Mathias Forcing

Mathias forcing $\mathbb{M}$ has conditions $\langle a, X\rangle$, where $a \in [\omega]^{<\omega}$, $X \in [\omega]^{\omega}$, and $\max(a) < \min(X)$.

$\langle b, Y\rangle \le \langle a, X\rangle$ iff $b \sqsupseteq a$, $Y \subseteq X$, and $b \setminus a \subseteq X$.

Mathias forcing is equivalent to forcing using the basic open sets in the Ellentuck space, ordered by $\subseteq$.

Connections with Forcing and Ultrafilters

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Ramsey ultrafilters have **complete combinatorics**. One way to state this is that if there is a supercompact cardinal in V, then any Ramsey ultrafilter in V is generic for the forcing $([\omega]^\omega, \subseteq^*)$ over the Solovay model $L(\mathbb{R})$.

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$\mathbb{M}$ is forcing equivalent to $\mathcal{P}(\omega)/\mathrm{Fin} * \mathbb{M}_{\dot{\mathcal{U}}}$, where $\mathcal{U}$ is the Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\mathrm{Fin}$.

Key properties from the Ellentuck space can be abstracted to give a general notion of a **topological Ramsey space**.

Abstract Topological Ramsey Spaces $(\mathcal{R}, \leq, r)$

$\mathcal{R}$ is a set. $\leq$ is quasi-order on $\mathcal{R}$.

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Basic open sets: $[a, X] = \{Y \in \mathcal{R} : a \sqsubset Y \leq X\}$.

The topology on $\mathcal{R}$ generated by the basic open sets is a refinement of the 'metric topology' on $\prod_{n<\omega} \mathcal{AR}_n$.

The Axioms **A.1** - **A.4**

A.1 (Sequencing)

1. $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.
2. $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.
3. $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

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A.2 (Finitization) There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that

1. $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,
2. $A \leq B$ iff $(\forall n)(\exists m)\ r_n(A) \leq_{\mathrm{fin}} r_m(B)$,
3. $\forall a, b, c \in \mathcal{AR}\ [a \sqsubset b \wedge b \leq_{\mathrm{fin}} c \rightarrow \exists d \sqsubset c\ a \leq_{\mathrm{fin}} d]$.

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$\mathrm{depth}_B(a)$ is the least n (if it exists) such that $a \leq_{\mathrm{fin}} r_n(B)$.
If $\mathrm{depth}_B(a) = n$, then $[\mathrm{depth}_B(a), B]$ denotes $[r_n(B), B]$.

A.3 (Amalgamation)

1. If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.
2. $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

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For $a \in \mathcal{AR}$, $|a|$ denotes the k such that $a \in \mathcal{AR}_k$.

If $n > |a|$, then $r_n[a, A] = \{r_n(X) : X \in [a, A]\}$.

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A.4 (Pigeonhole) Given $B \in \mathcal{R}$ and $a \in \mathcal{AR}$ with $\mathrm{depth}_B(a) < \infty$, then for any $\mathcal{O} \subseteq r_{|a|+1}[a, B]$, there is $A \in [\mathrm{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

Abstract Ellentuck Theorem

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Def. (Todorcevic) A triple $(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** if every subset of $\mathcal{R}$ with the Baire property is Ramsey, and if every meager subset of $\mathcal{R}$ is Ramsey null.

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Abstract Ellentuck Thm. (Todorcevic) If $(\mathcal{R}, \leq, r)$ satisfies Axioms **A.1 - A.4** and $\mathcal{R}$ is closed (in $\mathcal{AR}^{\mathbb{N}}$), then $(\mathcal{R}, \leq, r)$ is a topological Ramsey space.

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Motivating new Ramsey theorems.

Standard Examples of Topological Ramsey Spaces

1. Ellentuck space
2. Carlson-Simpson space of equivalence relations on ω with infinitely many equivalence classes (dual Ramsey)
3. Pröml-Voigt spaces of parameter words and ascending parameter words
4. Milliken space of block sequences $\mathrm{FIN}_k^{[\infty]}$
5. Carlson's space of infinite dimensional vector spaces $F^{\mathbb{N}}$ where F is a finite field.

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D. and Mijares have an example schema which encompasses (2) - (5) as special cases.

All of these spaces (except the Ellentuck space) have $\leq$ essentially given by a composition operator and are *surjective* spaces.

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Ultrafilters forced by topological Ramsey spaces

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$(\mathcal{R}, \leq^*)$ forces a maximal generic filter on $\mathcal{R}$.

The generic ultrafilter usually induces an ultrafilter on the base set $\mathcal{AR}_1$.

Selective Coideals and Complete Combinatorics

Given a topological Ramsey space $(\mathcal{R}, \le, r)$, a coideal $\mathcal{U} \subseteq \mathcal{R}$ is **selective** if for each $A \in \mathcal{U}$ and any collection $(A_a)_{a \in \mathcal{AR}|A}$ of members of $\mathcal{U} \restriction A$, there is a $U \in \mathcal{U}$ which diagonalizes $(A_a)_{a \in \mathcal{AR}|A}$.

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Thm. (Mijares) each $(\mathcal{R}, \leq)$ is forcing equivalent to forcing first with $(\mathcal{R}, \leq^*)$ to obtain a generic ultrafilter $\mathcal{U}$, and then forcing with the σ-closed localized version $(\mathcal{R}_{\mathcal{U}}, \leq)$ where the tails are in $\mathcal{U}$.

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Thm. (Di Prisco/Mijares/Nieto) In the presence of a supercompact cardinal, every selective coideal $\mathcal{U} \subseteq \mathcal{R}$ is generic for $(\mathcal{R}, \leq^*)$ over $L(\mathbb{R})$.

Laflamme's forcing to add a weakly Ramsey ultrafilter

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$\mathbb{P}_1 = ([\omega]^\omega, \leq_1)$. For $X, Y \in [\omega]^\omega$, enumerating them in increasing order and in blocks of increasing size as

$X = \langle x_1^1, x_1^2, x_2^2, x_1^3, x_2^3, x_3^3, \ldots \rangle$ and $Y = \langle y_1^1, y_1^2, y_2^2, y_1^3, y_2^3, y_3^3, \ldots \rangle$,

then $Y \leq_1 X$ iff $\forall m\ \exists n$ such that $\{y_1^m, \ldots, y_m^m\} \subseteq \{x_1^n, \ldots x_n^n\}$.

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$\mathbb{P}_1 = ([\omega]^\omega, \leq_1)$. For $X, Y \in [\omega]^\omega$, enumerating them in increasing order and in blocks of increasing size as

$X = \langle x_1^1, x_1^2, x_2^2, x_1^3, x_2^3, x_3^3, \dots \rangle$ and $Y = \langle y_1^1, y_1^2, y_2^2, y_1^3, y_2^3, y_3^3, \dots \rangle$, then $Y \leq_1 X$ iff $\forall m \, \exists n$ such that $\{y_1^m, \dots, y_m^m\} \subseteq \{x_1^n, \dots x_n^n\}$.

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So $(\mathbb{P}_1, \leq_1^*)$ is like $([\omega]^\omega, \subseteq^*)$ except the partial ordering is more restrictive.

Thm. (Laflamme) $(\mathbb{P}_1, \leq_1^*)$ forces a weakly Ramsey ultrafilter.

$\mathcal{U}$ is **weakly Ramsey** if for each finitary coloring c of $[\omega]^2$, there is a $U \in \mathcal{U}$ for which c takes on at most two colors on $[U]^2$.

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If κ is a Mahlo cardinal and G is Levy(κ)-generic over V, then any ultrafilter $\mathcal{U}$ on ω in $V[G]$ which is not Ramsey but is rapid and satisfies $\mathrm{RP}(k)$ for all k is generic over $\mathrm{HOD}(\mathbb{R})^{V[G]}$ (the original form of 'complete combinatorics').

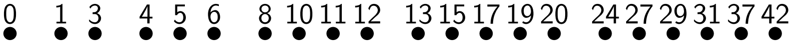
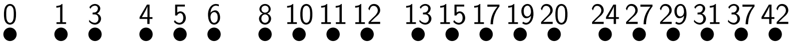


Figure: $X \in [\omega]^\omega$

Laflamme's forcing $(\mathbb{P}_1, \leq_1)$. Example: $Y \leq_1 X$



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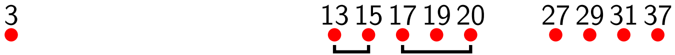

Figure: $Y \in [\omega]^\omega$ & $Y \leq_1 X$

The topological Ramsey space dense in $(\mathbb{P}_1, \leq_1)$

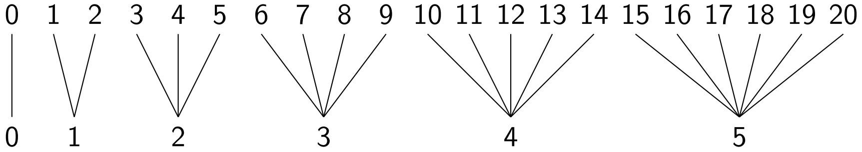

Figure: The maximum member of $\mathcal{R}_1$.

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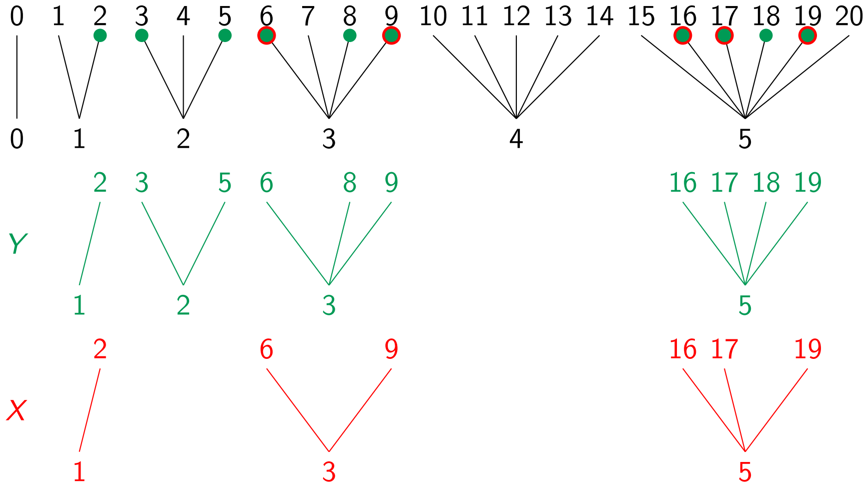

Figure: Two members X and Y of $\mathcal{R}_1$ with $X \leq Y$.

A subtree not in $\mathcal{R}_1$

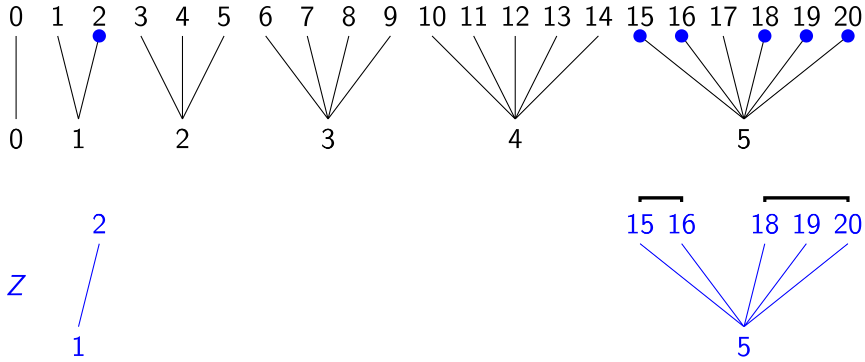

Figure: $Z \notin \mathcal{R}_1$

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Ramsey spaces $\mathcal{R}_\alpha$ dense inside Laflamme's forcings $\mathbb{P}_\alpha$, $\alpha < \omega_1$, were also constructed in [D./Todorcevic 15].

The n-square forcing of Blass

A subset of $\omega \times \omega$ of the form $s \times t$ is an ***n*-square** if $|s| = |t| = n$.

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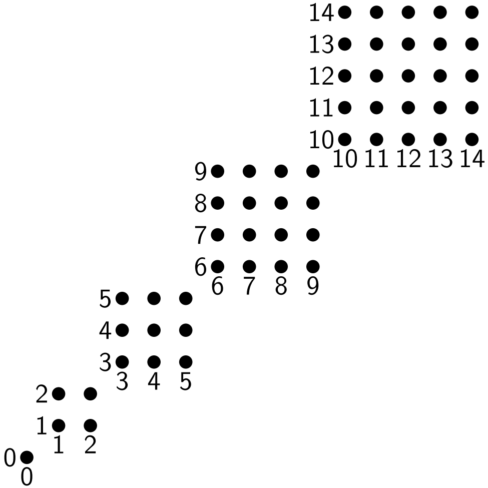
Def. $\mathcal{U}$ is a **p-point** if each sequence $X_0 \supseteq^* X_1 \supseteq^* \dots$ of members of $\mathcal{U}$ has a pseudointersection $U \in \mathcal{U}$; i.e. $U \subseteq^* X_i$ for all i.

The Ramsey space $\mathcal{H}_2$ dense in the n-square forcing



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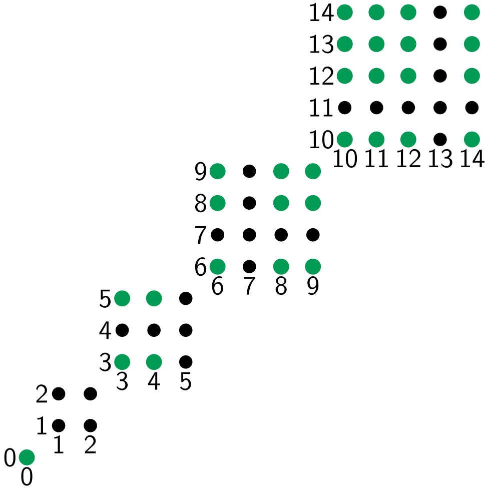

Figure: A member of X in $\mathcal{H}^2$.

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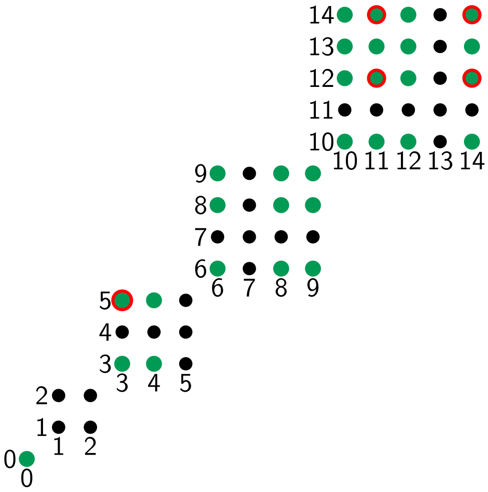

Figure: Two members Y and X of $\mathcal{H}^2$ with $Y \leq X$

The Hypercube topological Ramsey spaces

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Rem 2. Higher dimensional hypercube spaces were constructed in [D./Mijares/Trujillo] including a space where the dimension of the n-th block is $n + 1$.

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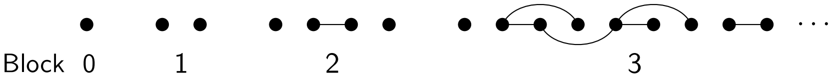

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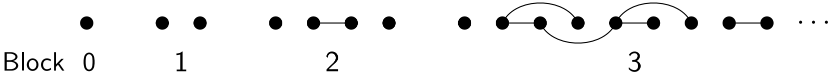

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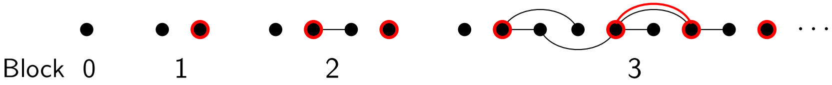

Figure: Example: Two members Y and X of $\mathcal{A}_3$ with $Y \leq X$

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The topological Ramsey space structure allows one to find formulas for the numbers d such that $\mathcal{U}_{\mathcal{K}} \to (\mathcal{U}_{\mathcal{K}})^m_{k,d}$, for such forced ultrafilters.

Topological Ramsey spaces dense in creature forcings

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In [D. TopApp 16], we proved that three of the examples of pure candidates for creature forcings given in [Roslanowski/Shelah 13] contain dense subsets which are topological Ramsey spaces.

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Thm. (Rosłanowski/Shelah) (CH) There is an ultrafilter $\mathcal{U}$ on base set $\mathcal{F}_{\mathbf{H}}$ generated by $\{\mathrm{pos}(\bar{t}_\alpha) : \alpha < \omega_1\}$ for a decreasing sequence of pure candidates $\langle \bar{t}_\alpha : \alpha < \omega_1 \rangle$ satisfying the partition theorem:

For any $\bar{t}$ such that $\mathrm{pos}(\bar{t}) \in \mathcal{U}$ and any partition of $\mathrm{pos}(\bar{t})$ into finitely many pieces, there is a pure candidate $\bar{s} \leq \bar{t}$ such that $\mathrm{pos}(\bar{s})$ is contained in one piece of the partition and $\mathrm{pos}(\bar{s}) \in \mathcal{U}$.

Remark 1. This generalizes the construction of a stable-ordered union ultrafilter on FIN using Hindman's Theorem.

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Remark 2. The proofs in [RS] use the Galvin-Glazer method extended to certain classes of creature forcings.

Thm. (D.) Three examples of sets of pure candidates in [RS] contain dense subsets forming topological Ramsey spaces.

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For two of these spaces, the pigeonhole principles rely on the following product tree Ramsey theorem.

New Product Tree Ramsey Theorem

Let $\prod_{j \in n+1, p} K_j = K_0 \times \cdots \times K_{p-1} \times [K_p]^k \times K_{p+1} \times \cdots \times K_n$.

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Thm. (D.) Given $k \geq 1$, a sequence of positive integers $(m_0, m_1, \dots)$, sets K_j, $j < \omega$ such that $|K_j| \geq j+1$, and a coloring

$$c : \bigcup_{n<\omega} \bigcup_{p \leq n} \prod_{j \in n+1, p} K_j \to 2,$$

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there are infinite sets $L, N \subseteq \omega$ such that $l_0 \leq n_0 < l_1 \leq n_1 < \dots$, and there are $H_j \subseteq K_j$, $j < \omega$, such that $|H_{l_i}| = m_i$ for each $i < \omega$, $|H_j| = 1$ for each $j \in \omega \setminus L$, and c is constant on

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Thm. (Di Prisco/Llopis/Todorcevic) There is an $R : (\mathbb{N}^+)^{<\omega} \to \mathbb{N}^+$ such that for every infinite sequence $(m_j)_{j<\omega}$ of positive integers and for every coloring

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there exist $H_j \subseteq R(m_0, \ldots, m_j)$, $|H_j| = m_j$, for $j < \omega$, such that c is constant on the product $\prod_{j\leq n} H_j$ for infinitely many $n < \omega$.

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Remark. The difference is that we need sets of size k to be able to move up and down indices of the product.

Conclusion for Day 1

Topological Ramsey spaces provide a unifying framework for many ultrafilters satisfying partition properties and, moreover, yield ultrafilters with complete combinatorics. They also motivate new Ramsey theorems.

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1. New canonical equivalence relations on fronts and barriers.
2. Exact Rudin-Keisler and Tukey structures as well as the structure of the Rudin-Keisler classes inside the Tukey types.

References for Day 1

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