Guessing sequences and indescribable cardinals

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Abstract

In this talk, we study club guessing sequences on indescribable cardinals. We explain outline of proofs of the following results:

① κ is indescribable and there is no club guessing sequences on the set $\{\alpha < \kappa$: α is regular$\}$.

② κ is indescribable and the ideal over κ defined by $\{X \subseteq \kappa : X$ does not carry a club guessing sequence$\}$ is locally κ^+-saturated.

We also show the consistency of the statement that κ is Π^1_1-indescribable and the Π^1_1-indescribable ideal over κ is κ^+-saturated.

Introduction

Throughout this talk,

κ : a regular uncountable cardinal

Reg : the class of regular cardinals.

Definition 1. $S \subseteq \kappa$: stationary set in κ.

A sequence $\langle a_\alpha : \alpha \in S \rangle$ is called a diamond sequence

$$\iff \forall A \subseteq \kappa,\ \exists \alpha \in S,\ a_\alpha = A \cap \alpha.$$

$\diamondsuit_\kappa(S) \iff$ there exists a diamond sequence on S.

Large cardinals and diamond principle

Fact 2 (Kunen-Jensen). *If κ is a subtle cardinal, then $\diamondsuit_\kappa(\mathrm{Reg} \cap \kappa)$ holds. Where κ is subtle*

$\iff$ $\forall C$, club in κ and $\forall$sequence $\langle a_\alpha : \alpha < \kappa \rangle$ with $a_\alpha \subseteq \alpha$, $\exists \alpha, \beta \in C$ with $\alpha < \beta$ and $a_\alpha = a_\beta \cap \alpha$.

Many reasonable large cardinals greater than Mahlo (e.g., measurable cardinal, Woodin cardinal, etc) are subtle.

κ: a suitable large cardinal $\Rightarrow \diamondsuit_\kappa(\mathrm{Reg} \cap \kappa)$ holds.

On the other hand, indescribable cardinals (Σ^m_n-indescribable cardinals, Π^m_n-indescribable cardinals) are not subtle in general.

Hence the above fact cannot be applied to indescribable cardinals:

Fact 3 (Woodin, Hauser). *Let m, n be natural numbers with $0 < m$.*

Con(ZFC+ $\exists\Pi^m_n$-indescribable (resp. Σ^m_n-indescribable) cardinal)

$\Rightarrow$ Con(ZFC+ $\exists\kappa$: Π^m_n-indescribable (resp. Σ^m_n-indescribable) and $\neg\diamondsuit_\kappa(\mathrm{Reg} \cap \kappa)$).

Club guessing sequence

Definition 4 (Shelah)**.** $S \subseteq \kappa$: stationary in κ.

A sequence $\vec{c} = \langle c_\alpha : \alpha \in A \rangle$ is called a (fully) club guessing sequence on A if

① $c_\alpha \subseteq \alpha$ is a club in α with $\mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha)$.

② For every club set C in κ, $\exists \alpha \in S, c_\alpha \subseteq C$.

$\mathrm{CG}_\kappa(S) \iff S$ carries a club guessing sequence.

A *tail club guessing* is defined by replacing $c_\alpha \subseteq C$ in ① by $c_\alpha \subseteq^* C$ (that is, $c_\alpha \setminus \xi \subseteq C$ for some $\xi < \alpha$).

Clearly $\diamondsuit_\kappa(S) \Rightarrow \mathsf{CG}_\kappa(S)$.

Fact 5 (Ishiu). *For stationary $S \subseteq \kappa$, $\mathsf{CG}_\kappa(S)$ if and only if S carries a tail club guessing sequence.*

Fact 6 (Shelah). *If $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$ for some regular $\mu < \kappa$ with $\mu^+ < \kappa$, then $\mathsf{CG}_\kappa(S)$ holds.*

Hence

① *If $\kappa = \mu^+$, then $\mathsf{CG}_\kappa(S)$ holds for every stationary subset S of $\{\alpha < \kappa : \mathrm{cf}(\alpha) < \mu\}$.*

② *If κ is weakly inaccessible and $S \subseteq \kappa \setminus \mathrm{Reg}$, then $\mathsf{CG}_\kappa(S)$ holds.*

So natural question is: how about the following cases?

① $\kappa = \mu^+$ and $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$.

② κ is weakly Mahlo and $S \subseteq \mathrm{Reg} \cap \kappa$.

Fact 7 (Shelah). ① *Con(ZFC* $+ \neg \mathrm{CG}_{\omega_1}(\omega_1)$*).*

② *Con(ZFC* $+ \neg \mathrm{CG}_\kappa(S)$ *for some* $\kappa = \mu^+$ *with* $\mu > \omega_0$ *and some stationary* $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$*).*

Then how about κ is weakly Mahlo and $S \subseteq \mathrm{Reg} \cap \kappa$?

Theorem 1

We prove the following theorem, which shows that indescribability of κ is not sufficient to ensure that $\mathrm{Reg} \cap \kappa$ carries a club guessing sequence.

Theorem 8. *Relative to certain large cardinal assumption, it is consistent that κ is Π^1_1-indescribable but $\mathrm{CG}_\kappa(\mathrm{Reg} \cap \kappa)$ fails.*

Club guessing ideals

Definition 9 (Shelah)**.** Let $S \subseteq \kappa$ and $\vec{c}$ be a tail club guessing sequence on S.

TCG($\vec{c}$) $:= \{X \subseteq \kappa : \exists C$ club in κ, $\forall \alpha \in X \cap A$, $c_\alpha \not\subseteq^* C\}$.

TCG($\vec{c}$) forms a normal ideal over κ.

Question: Can TCG($\vec{c}$) have good properties ?(e.g., saturation, precipitousness, etc.)

Fact 10 (Woodin, Ishiu). *Relative to certain large cardinal assumption, it is consistent that* $\kappa = \omega_1$ *and* $\mathsf{TCG}(\vec{C})$ *is* ω_2*-saturated for some tail club guessing sequence* $\vec{c}$.

Fact 11 (Foreman-Komjáth). *Relative to certain large cardinal assumption, it is consistent that* κ *is a successor caridnal* $> \omega_1$ *and* $\mathsf{TCG}(\vec{C})$ *is* κ^+*-saturated for some tail club guessing sequence* $\vec{c}$.

Non-Diamond ideals

Definition 12. $\mathrm{ND}_\kappa = \{X \subseteq \kappa : \neg\diamondsuit_\kappa(X)\}$.

ND_κ forms a normal ideal (but not necessary proper) over κ.

We consider the following variation of $\mathrm{TCG}(\vec{C})$ which is an analogue of ND_κ.

Definition 13. $\mathrm{NCG}_\kappa = \{X \subseteq \kappa : \neg\mathrm{CG}_\kappa(X)\}$.

$$\begin{aligned}\mathrm{NCG}_\kappa &= \{X \subseteq \kappa : X \text{ does not carry a tail club guessing}\} \\ &= \bigcap\{\mathrm{TCG}(\vec{c}) : \vec{c} \text{ is a tail club guessing sequence}\}.\end{aligned}$$

By Shelah's theorem,

① If $\kappa = \mu^+$, then

$$\mathrm{NCG}_\kappa | \{\alpha < \kappa : \mathrm{cf}(\alpha) < \mu\} = \mathrm{NS}_\kappa | \{\alpha < \kappa : \mathrm{cf}(\alpha) < \mu\}.$$

② If κ is weakly inaccessible, then

$$\mathrm{NCG}_\kappa | (\kappa \setminus \mathrm{Reg}) = \mathrm{NS}_\kappa | (\kappa \setminus \mathrm{Reg}).$$

Hence interesting ideals in this context are

$\mathrm{NCG}_\kappa | \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$ with $\kappa = \mu^+$,

and

$\mathrm{NCG}_\kappa | (\kappa \cap \mathrm{Reg})$ with κ being weakly Mahlo.

Saturation property of ideals

Fact 14. *Let $S \subseteq \kappa$ be such that $\diamondsuit_\kappa(S)$ holds. Then there are 2^κ-many almost disjoint stationary subsets of S. In particular $\mathsf{NS}_\kappa|S$ is not 2^κ-saturated.*

Lemma 15. *Let $S \subseteq \kappa$ be such that $\diamondsuit_\kappa(S)$ holds. Then there are 2^κ-many almost disjoint ND_κ-positive subsets of S. In particular,*

① $\mathsf{ND}_\kappa|S$ *is not 2^κ-saturated.*

② $\mathsf{NCG}_\kappa|S$ *is not 2^κ-saturated.*

κ: reasonable large cardinals $\Rightarrow \diamondsuit_\kappa(\mathsf{Reg} \cap \kappa)$

$\Rightarrow \mathsf{NS}_\kappa|\mathsf{Reg} \cap \kappa$ and $\mathsf{NCG}_\kappa|(\mathsf{Reg} \cap \kappa)$ are not saturated.

Fact 16 (Jech-Woodin). *Relative to a certain large cardinal assumption, it is consistent that κ is Mahlo and* $\mathrm{NS}_\kappa|(\mathrm{Reg} \cap \kappa)$ *is* κ^+*-saturated.*

Lemma 17. *Suppose that κ is* Π^1_1*-indescribable. Then* $\mathrm{NS}_\kappa|(\mathrm{Reg} \cap \kappa)$ *is not* κ^+*-saturated.*

A subset $X \subseteq \kappa$ is Π^m_n-indescribable (resp. Σ^m_n-indescribable) if $\forall R \subseteq V_\kappa$, $\forall \Pi^m_n$-formula (resp. Σ^m_n-formula) φ over the structure $\langle V_\kappa, \in, R\rangle$,

$$\langle V_\kappa, \in, R\rangle \models \varphi \Rightarrow \exists \alpha \in X, \langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$$

$\Pi_\kappa = \{X \subseteq \kappa : X$ is not Π^1_1-indescribable$\}$. Π_κ forms a normal ideal over κ.

Π_κ is also reffered as the weakly compact ideal over κ.

Fact 18. ① Reg $\cap\, \kappa \in \Pi^*_\kappa$.

② *Let $S \subseteq \kappa$ be a stationary set in κ. Then the set*

$$\{\alpha < \kappa : S \cap \alpha \text{ is stationary in } \alpha\}$$

lies in Π^*_κ.

Non-saturation of $\mathsf{NS}_\kappa|(\mathsf{Reg} \cap \kappa)$

Suppose that κ is Π^1_1-indescribable and $\mathsf{NS}_\kappa|(\mathsf{Reg}\cap\kappa)$ is κ^+-saturated. Then

① $\mathsf{NS}_\kappa|(\mathsf{Reg} \cap \kappa) \subseteq \Pi_\kappa$.

② $\Pi_\kappa = \mathsf{NS}_\kappa|S$ for some stationary $S \subseteq \mathsf{Reg}\cap\kappa$, because $\mathsf{NS}_\kappa|\mathsf{Reg}\cap\kappa$ is saturated.

③ $\{\alpha \in S : S \cap \alpha$ is stationary in $\alpha\} \in \Pi^*_\kappa = (\mathsf{NS}_\kappa|S)^*$,

④ However $\{\alpha \in S : S \cap \alpha$ is non-stationary in $\alpha\}$ is a stationary subset of S.

This is a contradiction.

Theorem 2

Hence almost all large cardinals greater than Mahlo refute the saturation property of $\mathsf{NS}_\kappa|(\mathsf{Reg} \cap \kappa)$.

On the other hand,

Theorem 19. *Relative to a certain large cardinal assumption, it is consistent that*

- κ *is* Π^1_1*-indescribable,*
- $\mathsf{CG}_\kappa(\mathsf{Reg} \cap \kappa)$ *holds, but*
- $\mathsf{NCG}_\kappa|(\mathsf{Reg} \cap \kappa)$ *is* κ^+*-saturated.*

Theorem 3

Theorem 20. *Suppose GCH and* κ *is a measurable cardinal. Then there exists a forcing extension in which the following hold:*

① κ *is* Π^1_1*-indescribable.*

② Π_κ *is* κ^+*-saturated.*

Corollary 21. *The following are equiconsistent:*

① *ZFC* $+ \exists$ *measurable cardinal.*

② *ZFC* $+ \kappa$ *is* Π^1_1*-indescribable* $+ \Pi_\kappa$ *is* κ^+*-saturated.*

Destroying club guessingness

Fix an inaccessible cardinal κ with $2^\kappa = \kappa^+$.

$\vec{c} = \langle c_\alpha : \alpha < \kappa \rangle$ is a club system $\iff$ $c_\alpha \subseteq \alpha$ is a club in α with $\mathrm{ot}(c_\alpha) = \mathrm{cf}(\alpha)$ for all $\alpha < \kappa$.

For a given club system, we define a poset which forces that the club system is never club guessing sequence on $\mathrm{Reg} \cap \kappa$.

Definition 22. $\mathbb{D}(\vec{c})$ is the set of all closed bounded subset p of κ with the property that $p \cap \alpha \not\subseteq c_\alpha$ for every $\alpha \in \mathrm{Reg} \cap \sup(p)$.

Lemma 23. $\mathbb{D}(\vec{c})$ *satisfies* κ^+*-c.c., and for every* $\mu < \kappa$, $\mathbb{D}(\vec{c})$ *has a* μ*-directed closed dense subset.*

Lemma 24. *Let* G *be a* $(V, \mathbb{D}(\vec{C}))$*-generic and* $C = \bigcup G$*. Then* C *is a club in* κ*, and* $c_\alpha \not\subseteq C$ *for every regular* $\alpha < \kappa$*.*

By a standard iteration of this poset, we can make any club system non-club guessing sequence on $\mathrm{Reg} \cap \kappa$. So the rest is to show that it preserves indescribability of κ.

Choose a sequence $\langle \dot{\vec{c}}(\xi) : \xi < \kappa^+ \rangle$ and $< \kappa$-support κ^+-stage iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \eta < \xi \leq \kappa^+ \rangle$ so that:

① $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\vec{c}}(\xi)$ is a club system".

② $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi = \mathbb{D}(\dot{\vec{c}}(\xi))$".

③ For every $\mathbb{P}_{\kappa^+}$-name $\dot{\vec{c}}$ of club system, the set

$$\{\xi < \kappa^+ : \Vdash_{\mathbb{P}_\xi} \text{“}\dot{\vec{c}}(\xi) = \dot{\vec{c}}\,\text{”}\}$$

is cofinal in κ^+.

④ For every $\xi \leq \kappa^+$, $\mathbb{P}_\xi$ satisfies the κ^+-c.c. and has μ-closed dense subset for all $\mu < \kappa$.

⑤ $\mathbb{P}_{\kappa^+}$ forces that $\neg \mathrm{CG}_\kappa(\mathrm{Reg} \cap \kappa)$.

Proof of Theorem 1

Assumption 25. *Suppose that* $\mathbb{P}_{\kappa^+}$ *forces the following:*

For every $(V, \mathbb{P}_{\kappa^+})$*-generic* G*, there exists an elementary embedding* $j : V \to M$ *(definable in* $V[G]$*) such that*

- *the critical point of* j *is* κ*.*
- M *is closed under* κ*-sequence in* $V[G]$*.*
- $\mathbb{P}_{\kappa^+}, G \in M$*.*

Remark 26. *Starting a model with measurable cardinal, we can construct such a model using a reverse Easton support iteration.*

For $\xi < \kappa^+$ and $(V, \mathbb{P}_{\kappa^+})$-generic G, let $G_\xi = G \cap \mathbb{P}_\xi$. G_ξ is $(V, \mathbb{P}_\xi)$-generic.

Lemma 27. $\mathbb{P}_{\kappa^+}$ *forces that:* $j : V \to M$ *can be extend to* $j : V[G_\xi] \to N$ *for some* $N \supseteq M$ *and* $\mathcal{P}(\kappa)^M = \mathcal{P}^N(\kappa)$.

Π^1_1-indescribability of κ in the generic extension follows from this Lemma:

Let G be a $(V, \mathbb{P}_{\kappa^+})$-generic and work in $V[G]$. Let $R \subseteq V_\kappa$ and φ be a Π^1_1-formula over $\langle V_\kappa, \in, R\rangle$.

Suppose $\langle V_\kappa, \in, R\rangle \vDash \varphi$. Then $\exists \xi < \kappa^+$,$R \in V[G_\xi]$. Because M is closed under κ-sequence, we have

$$M \vDash \text{``}\langle V_\kappa, \in, R\rangle \vDash \varphi\text{''}.$$

By the lemma, we can extend j to $j : V[G_\xi] \to N$ for some $N \supseteq M$ with $\mathcal{P}(\kappa)^M = \mathcal{P}^N(\kappa)$. Then

$$N \vDash \text{``}\langle V_\kappa, \in, R\rangle \vDash \varphi \text{ and } j(R) \cap V_\kappa = R\text{''}.$$

By the elementary of j,

$$V[G_\xi] \vDash \text{``}\exists \alpha < \kappa, \langle V_\alpha, \in, R \cap V_\alpha\rangle \vDash \varphi\text{''}.$$

so

$$V[G] \vDash \text{``}\exists \alpha < \kappa, \langle V_\alpha, \in, R \cap V_\alpha\rangle \vDash \varphi\text{''}.$$

Proof of Lemma

Because $\mathbb{P}_\xi$ adds no new $< \kappa$-sequence, we can identify

- $\dot{c}^\xi$ as a subset of $\kappa \times \mathcal{P}(\kappa) \times \mathbb{P}_\xi$.
- $p \in \mathbb{P}_\xi$ as a function with $\mathrm{dom}(p) \subseteq \xi$ and $|p| < \kappa$.

We use swapping coordinates arguments in Hauser's paper.

For $A \subseteq \kappa^+$, let

$$\mathbb{P}|A = \{p|A : p \in \mathbb{P}_{\kappa^+}\}.$$

Let π be a partial injection from κ^+ to κ^+. Then π induces a map $\mathbb{P}|\text{dom}(\pi)$ to $\mathbb{P}|\text{range}(\pi)$ such that :

- $\text{dom}(\pi(p)) = \pi``(\text{dom}(p))$,
- $\pi(p)(\xi) = p(\pi^{-1}(\xi))$ for $\xi \in \text{dom}(p)$.

By induction on $\xi < \kappa^+$, we take a good bijection π so that for each $\xi < \kappa^+$,

- $\dot{\vec{c}}(\pi(\xi)) = \{\langle \alpha, c, \pi(p) \rangle : \langle \alpha, c, p \rangle \in \dot{\vec{c}}(\xi)\}$.

Then π induces an isomorphism between $\mathbb{P}_{\kappa^+}$ to $\mathbb{P}_{\kappa^+}$.

Moreover we can require the following property for π:

- For every $p \in \mathbb{P}_\xi$ and $(V, \mathbb{P}_\xi)$-generic G_ξ with $p \in G_\xi$, there are $q \leq p$ such that $\pi(q) = q$ and $j``((\pi``G)_\xi)$ has a lower bound.

How to construct π

For $(V, \mathbb{P}_{\kappa^+})$-generic G and $\xi < \lambda^+$, let C_ξ be the ξ-th generic club induced by G.

For $\xi < \kappa^+$, suppose $\pi^* = \pi^{-1}|\xi$ is defined so that:

- π^* induces an isomorphism from $\mathbb{P}|\mathrm{dom}(\pi^*)$ to $\mathbb{P}_\xi$, and
- π^*"$G|(\mathrm{dom}(\pi^*))$ is $(V, \mathbb{P}_\xi)$-generic and j"$(\pi^*$"$G|(\mathrm{dom}\pi^*))$ has a lower bound.

There are cofinally many $\zeta < \kappa^+$ so that $\dot{\vec{c}}(\zeta) = \vec{c}(\xi)$. So we can choose $\xi^* < \kappa^+$ so that ξ_1 satisfies :

- $\dot{\vec{c}}(\xi^*) = \dot{\vec{c}}(\xi)$, and
- $C_\xi \in V[G_{\xi^*}]$.

Because C_{ξ^*} is generic over $V[G_{\xi^*}]$, we know that $C_{\xi^*} \neq j(\vec{c}(\xi))_\kappa$. Then we let $\pi(\xi^*) = \xi$.

If π can be taken as required, we can prove Lemma:

For given $p \in \mathbb{P}_\xi$, it is enough to find $(V, \mathbb{P}_{\kappa^+})$-generic G with the extension property.

Take an arbitrary $(V, \mathbb{P}_{\kappa^+})$-generic G^* with $p \in G^*$. Then we can choose $q \leq p$ such that $\pi(q) = q$ and $j``((\pi``G^*)_\xi)$ has a lower bound. $G := \pi``G^*$ is also $(V, \mathbb{P}_{\kappa^+})$-generic. Because $q = \pi(q) \in G$, we know $p \in G$. Since $j``G_\xi$ has a lower bouned, we can exetend j to $j : V[G_\xi] \to M[j(G_\xi)]$ for some $(M, j(\mathbb{P}_\xi))$-generic $j(G_\xi)$ with $j``G_\xi \subseteq j(G_\xi)$. This j and $M[j(G_\xi)]$ have requred properties.

Remarks

Remark 28. *Using Hauser's arguments carefully, we can show that :*

Con(ZFC+ $\exists\Pi^m_n$-indescribable (resp. Σ^m_n-indescribable) cardinal)

$\Rightarrow$ *Con(ZFC+ $\exists\kappa$: Π^m_n-indescribable (resp. Σ^m_n-indescribable) and* $\neg\mathrm{CG}_\kappa(\mathrm{Reg}\cap\kappa)$*).*

Remark 29. *A regular uncountable cardinal* κ *is* *strongly unfoldable* *if for every* $\lambda > \kappa$ *and every transitive model* M *of* ZFC^- *with* $|M| = \kappa \in M$*, there exist an transitive model* N *and elementary embedding* $j : M \to N$ *such that* $\mathrm{crit}(j) = \kappa$*,* $j(\kappa) > \lambda$*, and* $V_\lambda \subseteq N$*.*

Fact 30 (Džamonja-Hamkins). *Con(ZFC+* $\exists$ *strongly unfoldable cardinal)*

$\Rightarrow$ *Con(ZFC+* $\exists\kappa$*: strongly unfoldable cardinal and*

$\neg\diamondsuit_\kappa(\mathrm{Reg} \cap \kappa)$*).*

Corollary 31. *Con(ZFC+ $\exists$ strongly unfoldable cardinal)*

$\Rightarrow$ *Con(ZFC+ $\exists\kappa$: strongly unfoldable cardinal and*

$\neg \mathrm{CG}_\kappa(\mathrm{Reg} \cap \kappa)$*).*

Proof of Theorem 2

Let κ be an inaccesible with $2^\kappa = \kappa^+$.

- $\mathbb{Q}_0$ is a standard κ-closed poset which add a new club system, and
- $\mathbb{Q}_1$ is a standard κ-closed, κ^+-c.c. poset adds new club in κ which is almost contained in any club lies in the ground model.

Assumption 32. *Suppose that κ is inaccessible, $2^\kappa = \kappa^+$, and $\mathbb{Q}_0 * \mathbb{P}_{\kappa^+} * \mathbb{Q}_1$ forces the following:*

For every $\mathbb{Q}_0$-generic sequence $\vec{c}$, $(V[\vec{c}], \mathbb{P}_{\kappa^+})$-generic G, and $(V[\vec{c}, G], \mathbb{Q}_1)$-generic club C, there exists an elementary embedding $j : V[\vec{c}] \to M$ (definable in $V[\vec{c}, G, C]$) such that

- *the critical point of j is κ.*
- *M is closed under κ-sequence in $V[\vec{c}, G, C]$.*
- *$\mathbb{P}_{\kappa^+} \in M$ and $\vec{c}, G, C \in M$.*

We can construct a model which satisfies this assumption starting from the ground model with κ being measurable.

Under the above assumption, using Jech-Woodin's argument, we can find a good set $A \subseteq \kappa^+$ such that $\mathbb{P}|A$ is a complete suborder of $\mathbb{P}_{\kappa^+}$ and $\mathbb{Q}_0 * \mathbb{P}|A$ forces the following:

- $\vec{c}|(\mathrm{Reg} \cap \kappa)$ is a tail club guessing sequence on $\mathrm{Reg} \cap \kappa$.
- $\mathrm{NCG}_\kappa|(\mathrm{Reg} \cap \kappa) = \mathrm{TCG}(\vec{c})$ is κ^+-saturated.

Remark 33. *For $S \in \mathrm{NCG}_\kappa^+$, if $\mathrm{NCG}_\kappa|S$ is κ^+-saturated then there is a tail club guessing sequence $\vec{c}$ with $\mathrm{NCG}_\kappa|S = \mathrm{TCG}(\vec{c})$.*

Unfortunately, however, $\mathbb{Q}_0 * \mathbb{P}_A$ does not force that κ is Π^1_1-indescribable in general.

Proposition 34. *Suppose that GCH and κ is a measurable cardinal with Mitchell order 2. Then there exists a forcing extension in which the following hold:*

- *κ is Π^1_1-indescribable (actually Π^2_1-indescribable).*
- $\mathrm{NCG}_\kappa | (\mathrm{Reg} \cap \kappa)$ *is κ^+-saturated.*

Proposition 35. *Suppose that GCH and κ is an ω-strong cardinal. Then there exists a forcing extension in which the following hold:*

- κ *is totally indescribable.*
- $\mathsf{NCG}_\kappa|(\mathsf{Reg} \cap \kappa)$ *is κ^+-saturated.*

Strategically closedness

Definition 36. For a poset $\mathbb{P}$ and an ordinal α, $\Gamma_\alpha(\mathbb{P})$ denotes the following two players game:

$$\begin{array}{lclllllllll} I & : & p_0 & & p_1 & & \cdots & & p_{\omega+1} & & \cdots \\ II & : & & q_0 & & q_1 & \cdots & q_\omega & & p_{\omega+1} & \cdots \end{array}$$

Players choose elements of $\mathbb{P}$ alternately as $p_0 \leq q_0 \leq p_1 \cdots$. But at each limit stages, only player II moves.

II wins in $\Gamma_\alpha(\mathbb{P}) \iff$ II can take q_ξ for every $\xi < \alpha$.

A poset $\mathbb{P}$ is α-strategically closed if Player II of $\Gamma_\alpha(\mathbb{P})$ has a winning strategy.

Definition 37. An ideal I over κ is α-strategically closed if the generic ultrapower poset $\langle I^+, \subseteq_I \rangle$ associated with I is α-strategically closed.

Lemma 38. *If I is a κ-strategically closed normal ideal over κ, then $\Pi_\kappa \subseteq I$. In particular κ is Π^1_1-indescribable.*

Theorem 3(revised)

Theorem 39. *Suppose GCH and κ is a measurable cardinal. Then there exists a forcing extension in which the following hold:*

① κ *is* Π^1_1*-indescribable.*

② Π_κ *is* κ^+*-saturated and* κ*-strategically closed.*

Destroying Π^1_1-indescribability

For $X \subseteq \kappa$, $\mathbb{N}(X)$ is the set of all bounded subsets p of κ such that:

① $\sup(p) \in p$, and

② $\forall \alpha \in X$, $p \cap \alpha$ is non-stationary in α.

Lemma 40. $\mathbb{N}(X)$ *satisfies the* κ^+*-c.c., and is* κ*-strategically closed.*

Lemma 41. *Let* G *be a* $(V, \mathbb{N}(X))$*-generic filter and* $S = \bigcup G$*. Then* S *is statioanry in* κ *and* $S \cap \alpha$ *is non-stationary for every* $\alpha \in X$*. Hence* X *is not* Π^1_1*-indescribable in* κ*.*

Replacing club shootings in Jech-Woodin's argument by adding locally non-reflecting stationary sets, we can construct a model in which the following hold: There exists a normal ideal I over κ such that

- I is κ-strategically closed and κ^+-saturated.
- For every $X \subseteq \kappa$, if $X \in I$ then there exists a stationary subset S of κ such that $\forall \alpha \in X, S \cap \alpha$ is non-stationary.

Then $I = \Pi_\kappa$ holds, hence it is a required model.

Remark 42. *In our model, κ is Π^1_1-indescribable but not Π^1_2-indescribable.*

Lemma 43. *If κ is Π^1_{n+1}-indescribable, then the Π^1_n-indescribable ideal over κ is not κ^+-saturated.*

Questions

① How is the exact consistency strength of the following statements?

❶ κ is Π^1_1-indescribable and $\mathrm{NCG}_\kappa|(\mathrm{Reg} \cap \kappa)$ is κ^+-saturated.

❷ κ is Π^2_1-indescribable and $\mathrm{NCG}_\kappa|(\mathrm{Reg} \cap \kappa)$ is κ^+-saturated.

❸ κ is totally indescribable and $\mathrm{NCG}_\kappa|(\mathrm{Reg} \cap \kappa)$ is κ^+-saturated.

② Can the Π^1_2-indescirable ideal be saturated? How about Π^1_n-indescrbable ideal for $n > 2$?