A model with no strongly separable MAD families

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Outline

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2. Some connections
3. The Proof
Basic Definitions

- We say that two infinite subsets $a$ and $b$ of $\omega$ are *almost disjoint or a.d.* if $a \cap b$ is finite.
- We say that a family $\mathcal{A} \subset [\omega]^\omega$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint.
- A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.
Basic Definitions

- We say that two infinite subsets $a$ and $b$ of $\omega$ are *almost disjoint* or *a.d.* if $a \cap b$ is finite.
- We say that a family $\mathcal{A} \subset [\omega]^\omega$ is *almost disjoint* or *a.d.* if its members are pairwise almost disjoint.
- A *Maximal Almost Disjoint family, or MAD family* is an infinite *a.d.* family that is not properly contained in a larger *a.d.* family.
- If $\mathcal{A} \subset [\omega]^\omega$ is an *a.d.* family, then $\mathcal{I}(\mathcal{A})$ denotes the ideal on $\omega$ generated by $\mathcal{A}$. 
Completely Separable MAD families

**Definition**

An almost disjoint family $\mathcal{A}$ is said to be completely separable if for any $b \in \mathcal{I}^+(\mathcal{A})$, there is an $a \in \mathcal{A}$ with $a \subset b$. 

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Definition

An almost disjoint family $\mathcal{A}$ is said to be completely separable if for any $b \in I^+(\mathcal{A})$, there is an $a \in \mathcal{A}$ with $a \subset b$.

Their existence is connected to the existence of ADRs:

Definition

Given $\mathcal{C} \subset [\omega]^\omega$, we say that a family $\mathcal{A} = \{a_c : c \in \mathcal{C}\} \subset [\omega]^\omega$ is an almost disjoint refinement (ADR) of $\mathcal{C}$ if

1. $\forall c \in \mathcal{C} \ [a_c \subset c]$
2. $\forall c_0, c_1 \in \mathcal{C} \ [c_0 \neq c_1 \implies |a_{c_0} \cap a_{c_1}| < \omega]$. 
Completely Separable MAD families

Facts:

- If $\mathcal{C} \subset [\omega]^\omega$ has an ADR, then there is dense ideal $\mathcal{I}$ such that $\mathcal{I} \cap \mathcal{C} = 0$.

- $\mathcal{I}^+$ has an ADR for every dense $\mathcal{I}$ iff for every dense $\mathcal{I}$, there is a completely separable $\mathcal{A} \subset \mathcal{I}$.

- If $\mathcal{A}$ is completely separable, then for every $b \in \mathcal{I}^+(\mathcal{A})$, there are $\mathfrak{c}$ many $a \in \mathcal{A}$ such that $a \subset b$. 
Completely Separable MAD families

Facts:

- If $\mathcal{C} \subset [\omega]^{\omega}$ has an ADR, then there is dense ideal $\mathcal{I}$ such that $\mathcal{I} \cap \mathcal{C} = 0$.
- $\mathcal{I}^+$ has an ADR for every dense $\mathcal{I}$ iff for every dense $\mathcal{I}$, there is a completely separable $\mathcal{A} \subset \mathcal{I}$.
- If $\mathcal{A}$ is completely separable, then for every $b \in \mathcal{I}^+(\mathcal{A})$, there are $c$ many $a \in \mathcal{A}$ such that $a \subset b$.

Question (Erdos-Shelah)

Is there a completely separable MAD family $\mathcal{A} \subset [\omega]^{\omega}$? Is there a completely separable MAD $\mathcal{A} \subset \mathcal{I}$ for each dense $\mathcal{I}$?
Completely Separable MAD families

- Easy to see that answer is 'yes' if \( \alpha = \omega \)
- (Balcar, Simon, Vojtas): Yes if any one of these holds: \( s = \omega_1, b = \mathfrak{d} \), or \( \mathfrak{d} \leq \alpha \)
- (Balcar, Vojtas): Every non-principal ultrafilter has an ADR.
Completely Separable MAD families

- Easy to see that answer is 'yes' if $\alpha = \sigma$
- (Balcar, Simon, Vojtas): Yes if any one of these holds: $\sigma = \omega_1$, $b = d$, or $d \leq \alpha$
- (Balcar, Vojtas): Every non-principal ultrafilter has an ADR.

**Theorem (Shelah [2])**

*If $\sigma < \aleph_\omega$, then the answer is yes.*
There are several possible definitions. Let us rephrase the definition of completely separable MAD families:

**Definition**

*Given an ideal \( I \subset \mathcal{P}(\omega) \), let us say that a set \( A \subset [\omega]^{<\omega} \) is \( I \)-positive if for every \( a \in I \), \( \exists s \in A \ [s \cap a = 0] \).*

A set \( a \in \mathcal{P}(\omega) \) is in \( I^+ \) iff \( \{\{n\} : n \in a\} \) is \( I \)-positive.
There are several possible definitions. Let us rephrase the definition of completely separable MAD families:

**Definition**

*Given an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$, let us say that a set $A \subset [\omega]^{<\omega}$ is $\mathcal{I}$-positive if for every $a \in \mathcal{I}$, $\exists s \in A \ s \cap a = 0$. A set $a \in \mathcal{P}(\omega)$ is in $\mathcal{I}^+$ iff $\{\{n\} : n \in a\}$ is $\mathcal{I}$-positive.*

So $\mathcal{A}$ is completely separable iff for every $\mathcal{I}(\mathcal{A})$ positive set $A \subset [\omega]^{<\omega}$ consisting entirely of singletons, there are $\aleph_1$ many $a \in \mathcal{A}$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B = a$. 

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A model with no strongly separable MAD families
Strongly Separable MAD families

Steprans-Shelah definition: for every $\mathcal{I}(A)$ positive set $A \subset [\omega]^<\omega$, there are $\aleph_1$ many $a \in A$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

They applied this to the Calkin Algebra, $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$.

Definition: A masa in a $C^*$ algebra is a maximal, abelian, self adjoint subalgebra (i.e., $C^*$ subalgebra).

Theorem (Steprans and Shelah [3]): If there is a strongly separable MAD family (in their sense), then there is a masa in $\mathcal{C}(H)$ that is generated by its projections, and does not lift to a masa in $\mathcal{B}(H)$.

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Steprans-Shelah definition: for every $\mathcal{I}(\mathcal{A})$ positive set $A \subset [\omega]^{<\omega}$, there are $\mathfrak{c}$ many $a \in \mathcal{A}$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

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Strongly Separable MAD families

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A model with no strongly separable MAD families
Strongly Separable MAD families

Question (Steprans and Shelah)

Is there a strongly separable MAD family (in their sense)? Is there one with the property that for every \( I(\mathcal{A}) \) positive set \( A \subset [\omega]^{<\omega} \), there is at least one \( a \in \mathcal{A} \) such that there is an infinite pairwise disjoint \( B \in [A]^\omega \) so that \( \bigcup B \subset a \)? Can one be constructed if \( c < \aleph_\omega \)?
Strongly Separable MAD families

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Definition

An a.d. family \( \mathcal{A} \subset [\omega]^\omega \) is strongly separable if for every \( I(\mathcal{A}) \) positive \( A \subset [\omega]^{<\omega} \) there is an \( a \in \mathcal{A} \) such that there is an infinite \( B \in [A]^\omega \) so that \( \bigcup B \subset a \).
Strongly Separable MAD families

Question (Steprans and Shelah)

Is there a strongly separable MAD family (in their sense)? Is there one with the property that for every $\mathcal{I}(\mathcal{A})$ positive set $A \subset [\omega]^{<\omega}$, there is at least one $a \in \mathcal{A}$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B \subset a$? Can one be constructed if $\mathfrak{c} < \aleph_\omega$?

Definition

An a.d. family $\mathcal{A} \subset [\omega]^{\omega}$ is strongly separable if for every $\mathcal{I}(\mathcal{A})$ positive $A \subset [\omega]^{<\omega}$ there is an $a \in \mathcal{A}$ such that there is an infinite $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

Theorem (R.)

It is consistent that there are no strongly separable MAD families.
Metrizability of countable Fréchet groups

Definition
Recall that a topological space $X$ is Fréchet if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points in $A$ converging to $p$. 

Question (Malykhin)
Is it consistent that every countable Fréchet group is metrizable?
Metrizability of countable Fréchet groups

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Metrizability of countable Fréchet groups

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Question (Malykhin)
Is it consistent that every countable Fréchet group is metrizable?

Definition
Let us say that an ideal $\mathcal{I}$ is Fréchet if for every $\mathcal{I}$-positive $A \subset [\omega]^<\omega$, there is an infinite pairwise disjoint $B \in [A]^\omega$ so that $\forall a \in \mathcal{I} \ [|a \cap (\bigcup B)| < \omega]$. 

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If $\mathcal{I}$ is a Frechet ideal that is not countably generated, then we can define a non-metrizable Frechet topology on $\langle [\omega]^{<\omega}, \Delta \rangle$ by stipulating that

$$\{ A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega} \ [s \cap a = 0 \implies s \in A] \}$$

is a neighborhood base at 0.
Metrizability of countable Fréchet groups

- If $\mathcal{I}$ is a Frechet ideal that is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \triangle \rangle$ by stipulating that

$$\{ A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega} [s \cap a = 0 \implies s \in A] \}$$

is a neighborhood base at 0.

- The topology is Fréchet because a set $A \subset [\omega]^{<\omega}$ is $\mathcal{I}$ positive iff 0 is in the closure of $A$. And $A \subset [\omega]^{<\omega}$ has an subsequence converging to 0 iff there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that

$$\forall a \in \mathcal{I} \ [|a \cap (\bigcup B)| < \omega].$$
Metrizability of countable Fréchet groups

**Question (Gruenhage and Szeptycki)**

Is there an uncountable a.d. family $\mathcal{A} \subset [\omega]^\omega$ such that $I(\mathcal{A})$ is Fréchet?

Is there a Fréchet ideal $I \subset \mathcal{P}(\omega)$ that is not countably generated?

**Theorem (Brendle and Hrusak [1])**

It is consistent that no $I$ with fewer than $\mathfrak{c}$ generators is Fréchet.

My proof uses a modification of the forcing of Brendle and Hrusak.
The big picture:

- We assume $\diamondsuit(S_1^2)$ in the ground model and we do a finite support iteration of $\sigma$-centered forcings of length $\aleph_2$.

Given a strongly MAD family $A$ in the final model, there is a club of $\omega_1$ limits of $\omega_2$ where the maximality of $A$ reflects.

At a stage $\alpha$ when $A$ is maximal, we do a forcing that adds a set $A \subset [\omega]^\omega$ with the following two properties:

1. At no stage $\beta \geq \alpha$ is there an almost disjoint $A' \supset A$ so that $A$ is not $I(A')$ positive.
2. At no stage $\beta \geq \alpha$ is there an almost disjoint $A' \supset A$ so that there are $a \in A'$ and an infinite set $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

The second requirement was met by the Brendle-Hrusak forcing.
The Proof

The big picture:

- We assume $\diamondsuit(S^2_1)$ in the ground model and we do a finite support iteration of $\sigma$-centered forcings of length $\aleph_2$.
- Given a strongly MAD family $\mathcal{A}$ in the final model, there is a club of $\omega_1$ limits of $\omega_2$ where the maximality of $\mathcal{A}$ reflects.

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The Proof

The big picture:

- We assume $\Diamond(S_1^2)$ in the ground model and we do a finite support iteration of $\sigma$-centered forcings of length $\aleph_2$.

- Given a strongly MAD family $\mathcal{A}$ in the final model, there is a club of $\omega_1$ limits of $\omega_2$ where the maximality of $\mathcal{A}$ reflects.

- At a stage $\alpha$ when $\mathcal{A}$ is maximal, we do a forcing that adds a set $A \subset [\omega]^{<\omega}$ with the following two properties:
  - At no stage $\beta \geq \alpha$ is there an almost disjoint $\mathcal{A}' \supset \mathcal{A}$ so that $A$ is not $I(\mathcal{A}')$ positive.
  - At no stage $\beta \geq \alpha$ is there an almost disjoint $\mathcal{A}' \supset \mathcal{A}$ so that there are $a \in \mathcal{A}'$ and infinite set $B \in [A]^\omega$ so that $\bigcup B \subset a$.

The second requirement was met by the Brendle-Hrusak forcing.
The Proof

Suppose $\mathcal{A}$ is a MAD family (something weaker than maximality suffices). Then $I^+(\mathcal{A})$ is a selective coideal. Choose a selective ultrafilter $\mathcal{U} \subset I^+(\mathcal{A})$.

For $s \in \text{FIN} = [\omega]^{\omega} \setminus \{0\}$, cone$(s) = \{ t \in \text{FIN} : s \subset t \}$. We define

$$G = \{ A \subset \text{FIN} : \forall b \in \mathcal{U} \exists s \in \text{FIN}(b) [\text{cone}(s) \subset A] \}.$$ 

It is easy to check that $G$ is a filter on FIN. The forcing is $\mathbb{P} = \mathbb{L}(G)$. 
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- Suppose $\mathcal{A}$ is a MAD family (something weaker than maximality suffices). Then $I^+(\mathcal{A})$ is a selective coideal. Choose a selective ultrafilter $\mathcal{U} \subset I^+(\mathcal{A})$.

- For $s \in \text{FIN} = [\omega]^{\omega} \setminus \{0\}$, cone $(s) = \{ t \in \text{FIN} : s \subset t \}$. We define

$$G = \{ A \subset \text{FIN} : \forall b \in \mathcal{U} \exists s \in \text{FIN} (b) [\text{cone} (s) \subset A] \}.$$ 

- It is easy to check that $G$ is a filter on FIN. The forcing is $\mathbb{P} = \mathbb{L}(G)$.

- $\mathbb{P}$ adds a sequence $X : \omega \rightarrow \text{FIN}$. By genericity $\text{ran} (X)$ is $I(\mathcal{A})$ positive and for all $a \in \mathcal{A}$, $\forall \infty n \in \omega [X(n) \not\subset a]$. 

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The Proof

We also need to show:

- For all $a_0, \ldots, a_k \in \mathcal{A}$, and for all $b \in \mathbf{V}[G]$ with the property that for any $n \in \omega$, if $X(n) \cap (a_0 \cup \cdots \cup a_k) = 0$, then $b \cap X(n) \neq 0$, there is $a \in \mathcal{A}$ such that $|b \cap a| = \omega$

- For all infinite $B \in [\text{ran}(X)]^\omega$ that is in $\mathbf{V}[G]$, there is $a \in \mathcal{A}$ such that $|a \cap (\bigcup B)| = \omega$. 

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We also need to show:

- For all \( a_0, \ldots, a_k \in \mathcal{A} \), and for all \( b \in \mathcal{V}[G] \) with the property that for any \( n \in \omega \), if \( X(n) \cap (a_0 \cup \cdots \cup a_k) = 0 \), then \( b \cap X(n) \neq 0 \), there is \( a \in \mathcal{A} \) such that \( |b \cap a| = \omega \).

- For all infinite \( B \in [\text{ran}(X)]^\omega \) that is in \( \mathcal{V}[G] \), there is \( a \in \mathcal{A} \) such that \( |a \cap (\bigcup B)| = \omega \).

- We also need to make sure that these properties are preserved by the iteration. For this we need to strengthen the properties.
The Proof

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- For all $a_0, \ldots, a_k \in \mathcal{A}$, and for all $b \in \mathcal{V}[G]$ with the property that for any $n \in \omega$, if $X(n) \cap (a_0 \cup \cdots \cup a_k) = 0$, then $b \cap X(n) \neq 0$, there is $a \in \mathcal{A}$ such that $|b \cap a| = \omega$.
- For all infinite $B \in [\text{ran}(X)]^\omega$ that is in $\mathcal{V}[G]$, there is $a \in \mathcal{A}$ such that $|a \cap (\bigcup B)| = \omega$.
- We also need to make sure that these properties are preserved by the iteration. For this we need to strengthen the properties.

Lemma

Suppose $\{\hat{A}_n : n \in \omega\} \subset \mathcal{V}^\mathcal{P}$ so that for each $n \in \omega$, $\models \hat{A}_n \in [\text{ran}(\check{X})]^\omega$. Then there is $a \in \mathcal{A}$ such that for all $n \in \omega$, $\models |a \cap (\bigcup \hat{A}_n)| = \omega$. 

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The Proof

Lemma

Let \( \{\hat{a}_n : n \in \omega\} \subset \mathbf{V}^P \) and let \( a_0, \ldots, a_k \in \mathcal{A} \). Assume that for all \( n, m \in \omega \),
\[
\models \hat{X}(m) \cap (a_0 \cup \cdots \cup a_k) = 0 \implies \hat{X}(m) \cap \hat{a}_n \neq 0.
\]
Then there is \( a \in \mathcal{A} \setminus \{a_0, \ldots, a_k\} \) such that for all \( n \in \omega \), \( \models |a \cap \hat{a}_n| = \omega \).
The Proof

Lemma

Let \( \{\dot{a}_n : n \in \omega\} \subset V^P \) and let \( a_0, \ldots, a_k \in \mathcal{A} \). Assume that for all \( n, m \in \omega \),
\[
\models \dot{X}(m) \cap (a_0 \cup \cdots \cup a_k) = 0 \implies \dot{X}(m) \cap \dot{a}_n \neq 0.
\]
Then there is \( a \in \mathcal{A} \setminus \{a_0, \ldots, a_k\} \) such that for all \( n \in \omega \),
\[
\models \left| a \cap \dot{a}_n \right| = \omega.
\]

The proof uses Ramsey theory.

Lemma

Let \( F : \text{FIN} \to \omega \) such that for every \( s \in \text{FIN} \), \( F(s) \in s \). Then there is a set \( b \in [\omega]^{\omega} \) such that either (1) or (2) holds:

1. \( \forall s \in \text{FIN} \left( b \right) \forall c \in [b/s]^{\omega} \exists t \sqsupset c \left[ t \neq 0 \land F(s \cup t) \in t \right] \)
2. \( \forall c \in [b]^{\omega} \exists s \sqsupset c \left[ s \neq 0 \land \forall t \in [b/s]^{\omega} \left[ F(s \cup t \in s) \right] \right] \).
The Proof

Put $b = \omega \setminus (a_0 \cup \cdots \cup a_k) \in \mathcal{U}$. 
Put \( b = \omega \setminus \left( a_0 \cup \cdots \cup a_k \right) \in \mathcal{U} \).

Fix \( \sigma \in (\text{FIN})^{<\omega} \) and \( n \in \omega \). We may define a function
\[
F_{\langle \sigma, n \rangle} : \text{FIN}(b) \to b
\]
by \( F_{\langle \sigma, n \rangle}(s) \) is the least \( k \in s \) so that
\[
\neg \exists q \in \mathcal{P} \left[ \text{stem}(q) = \sigma \vdash \langle s \rangle \land q \downarrow k \notin \hat{a}_n \right]
\]
The Proof

- Put \( b = \omega \setminus (a_0 \cup \cdots \cup a_k) \in \mathcal{U} \).
- Fix \( \sigma \in (\text{FIN})^{<\omega} \) and \( n \in \omega \). We may define a function \( F_{\langle \sigma, n \rangle} : \text{FIN} \{b\} \rightarrow b \) by \( F_{\langle \sigma, n \rangle}(s) \) is the least \( k \in s \) so that \( \neg \exists q \in \mathcal{P} \left[ \text{stem}(q) = \sigma \cap \langle s \rangle \wedge q \vDash k \notin \hat{A}_n \right] \).
- We can find \( b_{\langle \sigma, n \rangle} \in \mathcal{U} \cap [b]^\omega \) which satisfies either (1) or (2).
- There is \( a \in \mathcal{A} \setminus \{a_0, \ldots, a_k\} \) so that \( \forall \langle \sigma, n \rangle \left[ |a \cap b_{\langle \sigma, n \rangle}| = \omega \right] \). This is the \( a \) we want.
Put $b = \omega \setminus (a_0 \cup \cdots \cup a_k) \in \mathcal{U}$.

Fix $\sigma \in (\text{FIN})^{<\omega}$ and $n \in \omega$. We may define a function $F_{\langle \sigma, n \rangle} : \text{FIN} (b) \to b$ by $F_{\langle \sigma, n \rangle} (s)$ is the least $k \in s$ so that

$\neg \exists q \in \mathcal{P} [\text{stem} (q) = \sigma \dashv \langle s \rangle \land q \vdash k \notin \hat{a}_n]$.

We can find $b_{\langle \sigma, n \rangle} \in \mathcal{U} \cap [b]^\omega$ which satisfies either (1) or (2).

There is $a \in \mathcal{A} \setminus \{ a_0, \ldots, a_k \}$ so that $\forall \langle \sigma, n \rangle [|a \cap b_{\langle \sigma, n \rangle}| = \omega]$. This is the $a$ we want.

Suppose, for a contradiction, that there is $n \in \omega$, and $p \in \mathcal{P}$ and $m \in \omega$ so that $p \vdash a \cap \hat{a}_n \subset m$. Put $\sigma = \text{stem} (p)$. 
A further modification of the approach will prove the consistency of “for every uncountable a.d. family \( \mathcal{A} \), \( \mathcal{I}(\mathcal{A}) \) is not Fréchet”.
Bibliography

