Lévy collapse and the precipitousness of normal ideals over $\omega_1$

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1. Introduction
Thm. (Foreman-Magidor-Shelah)

Let $\mu$ be a supercompact cardinal. Then

$$V^\text{Col}(\omega_1, < \mu) \models \text{"NS}_{\omega_1} \text{ is precipitous".}$$

- $\text{Col}(\omega_1, < \mu)$ is the Lévy collapse forcing $\mu$ to be $\omega_2$. 
Ishiu proved that ideals naturally defined from tail club guessing sequences are also precipitous in $\forall^{\text{Col}}(\omega_1, <\mu)$.

**Def.**
Suppose that $A \subseteq \text{Lim}(\omega_1)$.
\begin{align*}
\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle
\end{align*}
is called a **tail club guessing (t.c.g.)** sequence on $A$ if

(i) $c_\alpha$ is unbounded subset of $\alpha$ of order-type $\omega$,

(ii) For any club $C \subseteq \omega_1$ there exists $\alpha \in A$ with $c_\alpha \subseteq^* C$, i.e. $c_\alpha \setminus \beta \subseteq C$ for some $\beta < \alpha$.

**Def.**
For a t.c.g. sequence $\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle$, let

\begin{align*}
\text{TCG}(\vec{c}) := \text{the set of all } A' \subseteq \omega_1 \text{ such that } \vec{c} \upharpoonright A' \text{ is not a t.c.g. sequence.}
\end{align*}

$(\vec{c} \upharpoonright A' = \langle c_\alpha \mid \alpha \in A \cap A' \rangle)$
• TCG(\(\tilde{c}\)) is a normal proper ideal over \(\omega_1\) for any t.c.g. sequence \(\tilde{c}\).

• \(\Diamond\) holds in \(V^{Col(\omega_1,\mu)}\). Hence there exists a t.c.g. sequence.

• If \(\tilde{c}\) is a t.c.g. sequence in \(V^{Col(\omega_1,\mu)}\), then \(TCG(\tilde{c})\) is not equal to a restriction of \(NS_{\omega_1}\).

**Thm.** (Ishiu)

Let \(\mu\) be a supercompact cardinal. Then

\[V^{Col(\omega_1,\mu)} \models "TCG(\tilde{c})\) is precipitous for any t.c.g. seq. \(\tilde{c}"."\]
Question
How about other ideals over $\omega_1$?
What kind of ideals become precipitous in $V^{\text{Col}}(\omega_1,<\mu)$?

I do not know much on this question yet.
In this talk I will present

- other examples of normal ideals which are precipitous in $V^{\text{Col}}(\omega_1,<\mu)$,
- an example of normal ideal which is not precipitous in $V^{\text{Col}}(\omega_1,<\mu)$.

In the course of presenting examples of precipitous ideals, I also introduce new reflection principles associated to ideals.
2. Examples of precipitous ideals
2.1 ideals which become equal to $\text{NS}_{\omega_1}$

The ideals below are equal to $\text{NS}_{\omega_1}$ in $V^{\text{Col}}(\omega_1,\mu)$. Hence they are precipitous in $V^{\text{Col}}(\omega_1,\mu)$:

- the set of all $A \subseteq \omega_1$ on which a t.c.g. sequence exists
- the set of all $A \subseteq \omega_1$ on which a $\Diamond$-sequence exists
2.2 ideals defined from guessing sequences

**example 1**: ideals defined from weak club guessing sequences

**Def.**
Suppose that $A \subseteq \text{Lim}(\omega_1)$.
$c = \langle c_\alpha \mid \alpha \in A \rangle$ is called a **weak club guessing (w.c.g.)** sequence on $A$ if

(i) $c_\alpha$ is an unbounded subset of $\alpha$ of order-type $\omega$,
(ii) For any club $C \subseteq \omega_1$ there exists $\alpha \in A$ such that $c_\alpha \cap C$ unbounded in $\alpha$.

**Def.**
For a w.c.g. sequence $c = \langle c_\alpha \mid \alpha \in A \rangle$, let

$$\text{WCG}(c) := \text{the set of all } A' \subseteq \omega_1 \text{ such that } c \upharpoonright A' \text{ is not a w.c.g. sequence.}$$
• WCG(\( \bar{c} \)) is a normal proper ideal over \( \omega_1 \) for any w.c.g. sequence \( \bar{c} \).

• There exists a w.c.g. sequence in \( V^{\text{Col}}(\omega_1, < \mu) \).

• If \( \bar{c} \) is a w.c.g. seq. in \( V^{\text{Col}}(\omega_1, < \mu) \), then WCG(\( \bar{c} \)) differs from restrictions of \( \text{NS}_{\omega_1} \) and TCG(\( \bar{c}' \)) for any t.c.g. seq. \( \bar{c}' \).

**Thm.**
Let \( \mu \) be a supercompact cardinal. Then

\[ V^{\text{Col}}(\omega_1, < \mu) \models \text{“WCG}(\bar{c}) \text{ is precipitous for any w.c.g. seq. } \bar{c}” \]
example 2: ideals defined from $\Diamond'$-sequences

For any $\Diamond$-sequence $\vec{b} = \langle b_\alpha \mid \alpha \in A \rangle$,

"the set of all $A' \subseteq \omega_1$ with $\vec{b} \upharpoonright A'$ not a $\Diamond$-sequence"

is not an ideal.

For the following variant of $\Diamond^-$ the above construction gives normal ideals:

**Def.**
Suppose that $A \subseteq \omega_1$.

If $\vec{N} = \langle N_\alpha \mid \alpha \in A \rangle$ is called a $\Diamond'$-sequence on $A$ if

(i) $N_\alpha$ is a countable transitive model of $\text{ZFC}^-$,

(ii) for any $B \subseteq \omega_1$ the set $\{ \alpha \in A \mid B \cap \alpha \in N_\alpha \}$ is stationary.

• $\Diamond' \iff \Diamond^-$ \iff $\Diamond$. 

**Def.**
For a $\Diamond'$-sequence $\vec{N} = \langle N_\alpha \mid \alpha \in A \rangle$, let

$$DMD(\vec{N}) := \text{the set of all } A' \subseteq \omega_1 \text{ such that } \vec{N} \mid A' \text{ is not a } \Diamond'\text{-sequence}.$$ 

- $DMD(\vec{N})$ is a normal proper ideal over $\omega_1$ for any $\Diamond'$-seq. $\vec{N}$.
- In $V^{Col(\omega_1,<\mu)}$ there exists a $\Diamond'$-sequence.
- $NS_{\omega_1}$, $TCG(\vec{c})$ and $WCG(\vec{c})$ are in fact presaturated in $V^{Col(\omega_1,<\mu)}$. But $DMD(\vec{N})$ cannot be presaturated.

**Thm.**
Let $\mu$ be a supercompact cardinal. Then

$$V^{Col(\omega_1,<\mu)} \models \text{""DMD(\vec{N}) is precipitous for any } \Diamond'\text{-seq. } \vec{N}"".$$
Outline of proof of precipitousness

Roughly, the precipitousness of NS$_{\omega_1}$, TCG($\vec{c}$), WCG($\vec{c}$) and DMD($\vec{N}$) can be proved in the same manner. First recall that the precipitousness of NS$_{\omega_1}$ follows from the following two facts on the stationary reflection principle (SR):

**Def.**

SR $\equiv$ For any set $W \supseteq \omega_1$ and any stationary $X \subseteq [W]^{\omega}$, there exists $W' \subseteq W$ such that

(i) $|W'| = \omega_1 \subseteq W$

(ii) $X \cap [W']^{\omega}$ is stationary in $[W']^{\omega}$.

**Fact** (Foreman-Magidor-Shelah)
If $\mu$ is a supercompact cardinal, then SR holds in $V^{Col(\omega_1, < \mu)}$.

**Fact** (Foreman-Magidor-Shelah)
SR implies the precipitousness of NS$_{\omega_1}$. 

Below let $I$ be one of $\text{NS}_{\omega_1}$, $\text{TCG}(\bar{c})$, $\text{WCG}(\bar{c})$ and $\text{DMD}(\bar{N})$.

The precipitousness of $I$ can be proved by generalizing the previous facts.

**Step 1:** Define a tower of ideals $\langle I_W \mid W \supseteq \omega_1 \rangle$ associated with $I$ so that the reflection principle with respect to this tower holds in $V^{\text{Col}(\omega_1, < \mu)}$.

**Step 2:** Prove that the reflection principle with respect to the tower implies the precipitousness of $I$. 
Step 1 is quite the same for each $I = \text{NS}_{\omega_1}$, $\text{TCG}(\vec{c})$, $\text{WCG}(\vec{c})$, $\text{DMD}(\vec{N})$. We only use the following property of $I$:

**Lem.1**
Every $\sigma$-closed forcing preserves $I$-positive sets.

For each $\sigma$-closed poset $P$ let $I^P$ denotes the ideal $I$ defined in $V^P$.

If $I = \text{NS}_{\omega_1}$, then $I^P$ is $\text{NS}_{\omega_1}^{V^P}$.

If $I = \text{DMD}(\vec{N})$, then $I^P$ is $\text{DMD}(\vec{N})^{V^P}$.

...
• For each set $W \supseteq \omega_1$ define $I_W$ as follows:

$$I_W := \text{the set of all } X \subseteq [W]^{\omega} \text{ such that}$$

$$\exists P: \text{\(\sigma\)-closed poset which forces } |W| = \omega_1,$$

$$\exists \pi: \omega_1 \to W: \text{surjection in } V^P,$$

$$\{\alpha \in \omega_1 | \pi{``}\alpha \in X} \in I^P.$$ 

Using Lem.1, we can easily prove the following:

**Lem.2**

$$I_W = \text{the set of all } X \subseteq [W]^{\omega} \text{ such that}$$

$$\forall P: \text{\(\sigma\)-closed poset which forces } |W| = \omega_1,$$

$$\forall \pi: \omega_1 \to W: \text{surjection in } V^P,$$

$$\{\alpha \in \omega_1 | \pi{``}\alpha \in X} \in I^P.$$ 

**Note**

If $I = \text{NS}_{\omega_1}$, then $I_W$ is the nonstationary ideal over $[W]^{\omega}$.
The lemmata below naturally follows from the definition and Lem.1 and 2. Below let $W$ be a set $\supseteq \omega_1$:

**Lemma 3**

$I_W$ is a normal ideal over $[W]^\omega$.

**Proof of normality**

Suppose $\{X_a \mid a \in W\} \subseteq I_W$.

Let $P$ be a $\sigma$-closed poset forcing $|W| = \omega_1$, and take a surjection $\pi : \omega_1 \to W$ in $V^P$.

Then in $V^P$,

$$\{ \alpha \in \omega_1 \mid \pi^{"}\alpha \in \bigtriangleup_{a \in W} X_a \} = \bigtriangleup_{\beta < \omega_1} \{ \alpha \in \omega_1 \mid \pi^{"}\alpha \in X_{\pi(\beta)} \} \in I^P.$$ 

Hence $\bigtriangleup_{a \in W} X_a \in I_W$. \qed
**Lem. 4**

$I_{\omega_1}$ is essentially equal to $I$.

**Lem. 5**

$\langle I_W \mid W \supseteq \omega_1 \rangle$ forms a tower of ideals, i.e. if $\omega_1 \subseteq W \subseteq W'$, then for each $X \subseteq [W]^\omega$,

$$X \in I_W \iff \{x' \in [W']^\omega \mid x' \cap W \subseteq X\} \in I_{W'}.$$ 

**Lem. 6**

Every $\sigma$-closed forcing preserves $I_W$-positive sets, i.e. for any $\sigma$-closed $P$ and any $W \supseteq \omega_1$, if $X \subseteq [W]^\omega$ is $I_W$-positive in $V$, then $X$ is $I_W^P$-positive in $V^P$. 
• From Lem.6 and the usual generic ultrapower argument, it follows that the reflection principle with respect to the tower holds in $V^{Col(\omega_1,<\mu)}$:

**Def.**

$\text{RP}(I) \equiv$ For any set $W \supseteq \omega_1$ and any $I_W$-positive $X \subseteq [W]^{\omega}$, there exists $W' \subseteq W$ such that

(i) $|W'| = \omega_1 \subseteq W$

(ii) $X \cap [W']^{\omega}$ is $I_{W'}$-positive.

**Note**  $\text{RP}(\text{NS}_{\omega_1})$ is equivalent to SR.

**Prop.7**
If $\mu$ is supercompact, then $\text{RP}(I)$ holds in $V^{Col(\omega_1,<\mu)}$. 
The following holds for each $I = \text{NS}_{\omega_1}, \text{TCG}(\vec{c}), \text{WCG}(\vec{c}), \text{DMD}(\vec{N})$:

**Prop. 8**

$\text{RP}(I)$ implies the precipitousness of $I$.

This can be shown by the catch-your-tail argument for each $I$. But the proof is slightly different from each other. I do not know a uniform proof of Prop.8.
3. Example of non-precipitous ideal
We show the following:

**Prop. 9**
If \( \mu \) is an inaccessible cardinal, then there exists a normal non-precipitous ideal over \( \omega_1 \) in \( V^{Col(\omega_1, < \mu)} \).

First note that

\[
Col(\omega_1, < \mu) \simeq Col(\omega_1, < \mu) \ast Add(\omega_1, \mu),
\]

where \( Add(\omega_1, \mu) \) is the countable support product of \( Add(\omega_1) (= <\omega_1 \omega_1) \) of length \( \mu \).

Moreover \( \mu = \omega_2 \) and CH holds in \( V^{Col(\omega_1, < \mu)} \).

Hence it suffices to show the following:

**Prop. 10**
Assume CH. Then there exists a normal non-precipitous ideal over \( \omega_1 \) in \( V^{Add(\omega_1, \omega_2)} \).
• We use canonical functions:

For each \( \eta < \omega_2 \), take a surjection \( \pi_\eta : \omega_1 \to \eta \), and define \( h_\eta : \omega_1 \to \omega_1 \) as

\[
h_\eta(\alpha) := \text{the order type of } \pi_\eta"\alpha.\]

We call \( h_\eta \) the canonical function for \( \eta \).

For any normal ideal \( J \) over \( \omega_1 \), \( h_\eta \) represents \( \eta \) in the generic ultrapower by \( J \).
Outline of Proof of Prop.10

Let $G$ be $\text{Add}(\omega_1, \omega_2)$-generic filter. We work in $M := V[G]$.

For each $\xi < \omega_2$ let $f_\xi : \omega_1 \rightarrow \omega_1$ be the $\xi$-th fnt. added by $G$.

For each $\xi, \eta < \omega_2$ let

$$A_{\xi, \eta} := \{ \alpha \in \omega_1 \mid f_\xi(\alpha) \leq h_\eta(\alpha) \},$$

and let

$$J := \text{the normal ideal generated by} \{ A_{\xi, \eta} \mid \xi, \eta < \omega_2 \}.$$

(In the generic ultrapower by $J$, each $f_\xi$ represents an ordinal which is greater than all $\eta < \omega_2^M$.)

Then we can prove the following:

- $J$ is a proper ideal.

- $\omega_2^M$ is not in the well-founded part of the generic ultrapower by $J$. □
4. Question
4.1 ideals defined from unbounded functions

Let $h_\eta$ be the canonical function for each $\eta < \omega_2$.

- In the proof of Prop.10, each $f = f_\xi$ has the property below:
  
  “For any $\eta < \omega_2$ the set $\{\alpha \in \omega_1 | f(\alpha) > h_\eta(\alpha)\}$ is stationary.”

  A (maybe partial) function $f : \omega_1 \to \omega_1$ with this property is said to be **unbounded**.

- $\Diamond$ implies the existence of unbounded functions.

**Def.**

For an unbounded function $f : \omega_1 \to \omega_1$, let

$$\text{UBD}(f) := \text{the normal ideal generated by } \{A_\eta | \eta < \omega_2\}$$

$$= \text{the set of all } A \subseteq \omega_1 \text{ with } f | A \text{ not unbounded.}$$

Here $A_\eta = \{\alpha \in \omega_1 | f(\alpha) \leq h_\eta(\alpha)\}$. 
- I do not know whether UBD(f) is precipitous or not in $V^{Col}(\omega_1, <\mu)$.

- As is DMD($\vec{N}$), UBD(f) cannot be presaturated.

- As in the case of NS$_{\omega_1}$, TCG($\vec{c}$), WCG($\vec{c}$) and DMD($\vec{N}$), we can construct a tower of ideals associated to UBD(f), and can prove that the reflection principle RP(UBD(f)) with respect to this tower holds in $V^{Col}(\omega_1, <\mu)$.

  But I do not know whether RP(UBD(f)) implies the precipitousness of UBD(f) or not.

**Question**

Is UBD(f) precipitous in $V^{Col}(\omega_1, <\mu)$?

Does RP(UBD(f)) imply the precipitousness of UBD(f)?
4.2 existence of non-precipitous ideals

It is consistent that every normal ideal over $\omega_1$ is precipitous. In fact, if $\text{NS}_{\omega_1}$ is saturated, then every normal ideal over $\omega_1$ is a restriction of $\text{NS}_{\omega_1}$ to some stationary set. Hence if $\text{NS}_{\omega_1}$ is saturated, then every normal ideal over $\omega_1$ is precipitous.

**Question**

In what situation normal non-precipitous ideals over $\omega_1$ exist?

For example, $\diamondsuit$ implies the existence of normal non-precipitous ideals?
Thank you very much.