All automorphisms of all Calkin algebras
Part II: Automorphisms of $\mathcal{C}_{\aleph_1}$

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Question (Brown–Douglas–Fillmore, 1977)

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As usually, Φ is *inner* if for some $u \in \mathcal{C}(H)$ we have

$$\Phi(a) = uau^*$$

for all a.

Proposition

An automorphism Φ *of the Calkin algebra is inner if and only if there is a *-homomorphism* $\Psi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ *such that the diagram*

$$\begin{CD} \mathcal{B}(H) @>{\Psi}>> \mathcal{B}(H) \\ @V{\pi}VV @VV{\pi}V \\ \mathcal{C}(H) @>>{\Phi}> \mathcal{C}(H) \end{CD}$$

commutes.

A rather complete picture

Theorem (Phillips–Weaver, 2006)

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What can be said in the case when H is nonseparable?

A sadly incomplete answer will take up today's and tomorrow's lectures.

Notation and the theorem

$\ell_2(\kappa)$: a complex Hibert space of character density κ.
$\mathcal{B}_\kappa = \mathcal{B}(\ell_2(\kappa))$: the algebra of bounded linear operators.
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Assume MA+TA. Then all automorphisms of $\mathcal{C}_{\aleph_1}$ are inner.

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Assume MA+TA. Then all automorphisms of $\mathcal{C}_{\aleph_1}$ are inner.

We really prove: If all automorphisms of $\mathcal{C}_{\aleph_0}$ are inner and MA holds, then all automorphisms of $\mathcal{C}_{\aleph_1}$ are inner.

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Fact

All automorphisms of A are inner iff $u \mapsto \mathrm{Ad}\, u$ is a surjection.

Reduction to the separable case: Notation

For $\xi < \aleph_1$ let
$\mathcal{B}_\xi = \ell_2(\xi)$
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$p_\xi = \mathrm{proj}_{\ell_2(\xi)}$.

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For a club $\mathbf{C} \subseteq \aleph_1$ let

$$\mathcal{D}[\mathbf{C}] = \{a \in \mathcal{B}_{\aleph_1} : ap_\xi = p_\xi a \text{ for all } \xi \in \mathbf{C}\}.$$

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Pick $M \prec H_{\mathfrak{c}^+}$ and let $\delta = M \cap \omega_1$.
If $a \in M \cap \mathcal{B}_{\aleph_1}$ then $ap_\delta = p_\delta a$.
If $a \in \mathcal{B}_{\aleph_1}$ and M_ξ, $\xi < \omega_1$, is an $\in$-chain of elementary submodels of $H_{\mathfrak{c}^+}$ such that $a \in M_0$, then with

$$\mathbf{C} = \{M_\xi \cap \aleph_1 : \xi < \aleph_1\}$$

we have that $a \in \mathcal{D}[\mathbf{C}]$. □

Representations

Fix $\Phi \in \mathrm{Aut}(\mathcal{C}_{\aleph_1})$.

Fix $\Phi_* \colon \mathcal{B}_{\aleph_1} \to \mathcal{B}_{\aleph_1}$ such that

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For each $\xi \in \mathbf{C}_\Phi$ pick v_ξ such that $\operatorname{Ad} v_\xi$ is a representation of $\Phi \restriction \mathcal{C}_\xi$.

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. . . but the truth is more interesting.

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or rather, for unitaries u, v in $\mathcal{B}(H)$, the relation

$$v \sim w \text{ iff } \operatorname{Ad}\pi(u) = \operatorname{Ad}\pi(v).$$

A description of $\sim$ on $\mathcal{B}(H)$

Lemma

For u and v in $\mathcal{U}(\mathcal{B}(H))$ we have $\operatorname{Ad} u = \operatorname{Ad} v$ if and only if $u = zv$ for some $z \in \mathbb{C}$.

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Lemma

For u and v in $\mathcal{B}(H)$ such that $\pi(u)$ and $\pi(v)$ are unitaries in $\mathcal{C}(H)$ we have $\operatorname{Ad} \pi(u) \equiv \operatorname{Ad} \pi(v)$ if and only if

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Then for all $\aleph_0 \leq \eta < \xi$ we have that

$$v_\eta - p_\eta v_\xi$$

is compact.

Let

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Fact

$T = \langle X_\xi, \pi_{\xi\eta} \colon \omega \leq \eta < \xi < \omega_1 \rangle$ is a Polish ω_1-tree.

Assume T has an ω_1-branch, w_ξ, for $\xi < \omega_1$.

Define $w \in \mathcal{B}(\ell_2(\aleph_1))$ by

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Then $\operatorname{Ad} \pi(w)$ implements Φ.
So we may assume T has no ω_1-branches.

The 'local' tree

For $a \in \mathcal{B}_{\aleph_1}$ define

$$X(a)_\xi = \{waw^* \colon w \in X_\xi\}$$

wrt the norm metric and $\pi_{\xi\eta} \colon X(a)_\xi \to X(a)_\eta$ via

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We add a generic $a \in \mathcal{B}_{\aleph_1}$ such that $T(a)$ has no ω_1-branches.

Adding a generic operator

Consider the forcing $\mathbb{P}_Z$ with conditions $p = (F_p, M_p)$, where $F_p \subseteq Z$ is finite and M_p is a matrix over $\mathbb{Q} + i\mathbb{Q}$ indexed by $F_p \times F_p$ such that $\|M_p\| < 1$.

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($\mathbb{P}_Z$ is essentially adding a Cohen real to the unit ball of $\mathcal{B}(\ell_2(Z))$ in the weak operator topology.)

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Bad news and good news

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Lemma

If T has no cofinal branch, then $\mathbb{P}$ forces that $T(a)$ has no cofinal branch.

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Since this is a contradiction, we conclude that T has a cofinal branch, hence Φ is inner. □

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In particular, it cannot have a cofinal special subtree.

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Next time

What we don't know.