All automorphisms of all Calkin algebras
Part I \( \frac{I}{II} \): The introduction

Ilijas Farah

York University

RIMS, Kyoto, November 2009
An automorphism \( \Phi \) of \( \mathcal{P}(\mathbb{N})/\text{Fin} \) is \textit{trivial} if there is \( h: \mathbb{N} \to \mathbb{N} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) & \overset{X \mapsto h^{-1}(X)}{\longrightarrow} & \mathcal{P}(\mathbb{N}) \\
\downarrow^{\pi_{\text{Fin}}} & & \downarrow^{\pi_{\text{Fin}}} \\
\mathcal{P}(\mathbb{N})/\text{Fin} & \overset{\Phi}{\longrightarrow} & \mathcal{P}(\mathbb{N})/\text{Fin}
\end{array}
\]

commutes.
Rigidity I

Theorem (W. Rudin, 1957)

$CH$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^c$ automorphisms
Rigidity I

Theorem (W. Rudin, 1957)

$CH$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^c$ automorphisms and only $c$ of them are trivial.
Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^\mathfrak{c}$ automorphisms and only $\mathfrak{c}$ of them are trivial.

Theorem (Shelah, 1979)

If ZFC is consistent then so is ZFC+‘All automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial.’
Theorem (W. Rudin, 1957)

$CH$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^\mathfrak{c}$ automorphisms and only $\mathfrak{c}$ of them are trivial.

Theorem (Shelah–Steprāns, 1989)

$PFA$ implies ‘All automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial.’
Theorem (W. Rudin, 1957)

$CH$ implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^\mathfrak{c}$ automorphisms and only $\mathfrak{c}$ of them are trivial.

Theorem (Velickovic, 1989)

$TA + MA$ implies ‘All automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial.’
Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\text{Fin}$ has $2^\mathfrak{c}$ automorphisms and only $\mathfrak{c}$ of them are trivial.

Theorem (Velickovic, 1989)

TA+MA implies ‘All automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial.’

TA:= Todorcevic’s Axiom (aka OCA)

Assume $X$ is a separable metric space and $K \subseteq [X]^2$ is open. Then either there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K$ or $X = \bigcup_{n<\omega} X_n$ so that $[X_n]^2 \cap K = \emptyset$ for all $n$. 
Rigidity beyond $\aleph_0$

Theorem (Velickovic, 1989)

$TA + MA$ implies all automorphisms of $P(\aleph_1)/\text{Fin}$ are trivial.
Rigidity beyond $\aleph_0$

Theorem (Velickovic, 1989)

TA + MA implies all automorphisms of $\mathcal{P}(\aleph_1)/\text{Fin}$ are trivial. PFA implies all automorphisms of $\mathcal{P}(\kappa)/\text{Fin}$ are trivial, for every infinite cardinal $\kappa$. 
Theorem (Shelah, 1979)
If ZFC is consistent then so is ZFC+'every autohomeomorphism of $\beta \aleph_0 \setminus \aleph_0$ has an extension to a continuous self-map of $\beta \aleph_0$.

Theorem (Farah, 1998)
TA+MA implies that if X is a 0-dimensional locally compact Polish space then every autohomeomorphism of $\beta X \setminus X$ has an extension to a continuous self-map of $\beta X$.
Stone duality

Boolean algebra \( \mathcal{P}(\mathbb{N}) \) \( \leftrightarrow \) Topological space \( \beta\mathbb{N} \)

\( \mathcal{P}(\mathbb{N}/\text{Fin}) \) \( \leftrightarrow \) \( \beta\mathbb{N} \setminus \mathbb{N} \)

**Theorem (Shelah, 1979)**

*If ZFC is consistent then so is ZFC + ‘every autohomeomorphism of \( \beta\mathbb{N} \setminus \mathbb{N} \) has an extension to a continuous self-map of \( \beta\mathbb{N} \).*
Stone duality

\[
\begin{array}{ccc}
\text{Boolean algebra} & \longleftrightarrow & \text{Topological space} \\
\mathcal{P}(\mathbb{N}) & \longleftrightarrow & \beta\mathbb{N} \\
\mathcal{P}(\mathbb{N}/\text{Fin}) & \longleftrightarrow & \beta\mathbb{N} \setminus \mathbb{N}
\end{array}
\]

**Theorem (Shelah, 1979)**

*If ZFC is consistent then so is ZFC+‘every autohomeomorphism of \( \beta\mathbb{N} \setminus \mathbb{N} \) has an extension to a continuous self-map of \( \beta\mathbb{N} \).*

**Theorem (Farah, 1998)**

*TA+MA implies that if \( X \) is a 0-dimensional locally compact Polish space then every autohomeomorphism of \( \beta X \setminus X \) has an extension to a continuous self-map of \( \beta X \).*
More rigidity

Conjecture (Farah, 2000)

*PFA implies that all isomorphisms* $\Phi: P(\mathbb{N})/\mathcal{I} \to P(\mathbb{N})/\mathcal{J}$ *are trivial, whenever* $\mathcal{I}$ *and* $\mathcal{J}$ *are analytic ideals.*
Conjecture (Farah, 2000)

PFA implies that all isomorphisms $\Phi: \mathcal{P}(\mathbb{N})/\mathcal{I} \rightarrow \mathcal{P}(\mathbb{N})/\mathcal{J}$ are trivial, whenever $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals.

Theorem (Farah, 1997–2004)

Confirmed the conjecture for many ideals, including all nonpathological ideals.
More rigidity

**Conjecture (Farah, 2000)**

*PFA implies that all isomorphisms $\Phi: \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ are trivial, whenever $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals.*

**Theorem (Farah, 1997–2004)**

*Confirmed the conjecture for many ideals, including all nonpathological ideals.*

**Corollary**

*PFA implies that quotients over two nonpathological ideals are isomorphic if and only if the ideals are isomorphic.*
Boolean groups

If

If $\mathcal{P}(\mathbb{N}) \xrightarrow{F} \mathcal{P}(\mathbb{N})$ commutes then we say $F$ is a representation of $\Phi$. 
Boolean groups

If

\[
\begin{array}{c}
P(\mathbb{N}) \xrightarrow{\pi_{\text{Fin}}} P(\mathbb{N}) \\
\downarrow \pi_{\text{Fin}} \quad \quad \quad \quad \quad \quad \quad \downarrow \pi_{\text{Fin}}
\end{array}
\]

\[
\begin{array}{c}
P(\mathbb{N})/\text{Fin} \xrightarrow{\Phi} P(\mathbb{N})/\text{Fin} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow
\end{array}
\]

commutes then we say \( F \) is a representation of \( \Phi \).

\[ (P(\mathbb{N}), \Delta) \cong (\mathbb{Z}/2\mathbb{Z})^\mathbb{N} \]
Boolean groups

If

\[
\begin{array}{c}
P(\mathbb{N}) \xrightarrow{F} P(\mathbb{N}) \\
& \downarrow \pi_{\text{Fin}} \quad \downarrow \pi_{\text{Fin}} \\
\mathcal{P}(\mathbb{N})/\text{Fin} \xrightarrow{\Phi} \mathcal{P}(\mathbb{N})/\text{Fin}
\end{array}
\]

commutes then we say \( F \) is a representation of \( \Phi \).

\((\mathcal{P}(\mathbb{N}), \Delta) \cong (\mathbb{Z}/2\mathbb{Z})^\mathbb{N}\)

A group isomorphism \( \Phi: \mathcal{P}(\mathbb{N})/I \rightarrow \mathcal{P}(\mathbb{N})/J \) is trivial if it has a representation that is a group homomorphism.
Theorem (Farah, 2000)

Every group isomorphism $\Phi: \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Borel-measurable representation is trivial whenever $\mathcal{J}$ is a nonpathological ideal.
Theorem (Farah, 2000)

Every group isomorphism $\Phi: \mathcal{P}(\mathbb{N})/\mathcal{I} \rightarrow \mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Borel-measurable representation is trivial whenever $\mathcal{J}$ is a nonpathological ideal.

Lemma

If $\mathcal{I}$ is a proper analytic ideal then $\mathcal{P}(\mathbb{N})/\mathcal{I} \cong \mathcal{P}(\mathbb{N})$ (as a group).
Theorem (Farah, 2000)

Every group isomorphism $\Phi: \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Borel-measurable representation is trivial whenever $\mathcal{J}$ is a nonpathological ideal.

Lemma

If $\mathcal{I}$ is a proper analytic ideal then $\mathcal{P}(\mathbb{N})/\mathcal{I} \cong \mathcal{P}(\mathbb{N})$ (as a group).

Proof.
Both are $\mathbb{F}_2$-vector spaces of dimension $2^{\aleph_0}$.
Gelfand–Naimark–Segal duality

For a compact Hausdorff space $X$ let $C(X)$ be the $C^*$-algebra of all continuous complex-valued functions on $X$. 
Gelfand–Naimark–Segal duality

For a compact Hausdorff space $X$ let $C(X)$ be the C*-algebra of all continuous complex-valued functions on $X$.

\[
\begin{align*}
\beta \mathbb{N} & \leftrightarrow \quad C(\beta \mathbb{N}) \\
\beta \mathbb{N} \setminus \mathbb{N} & \leftrightarrow \quad C(\beta \mathbb{N} \setminus \mathbb{N})
\end{align*}
\]
For a compact Hausdorff space $X$ let $C(X)$ be the C*-algebra of all continuous complex-valued functions on $X$.

\[ \beta \mathbb{N} \leftrightarrow C(\beta \mathbb{N}) \]
\[ \beta \mathbb{N} \setminus \mathbb{N} \leftrightarrow C(\beta \mathbb{N} \setminus \mathbb{N}) \]

**Theorem (Shelah, 1979)**

*If ZFC is consistent then so is ‘all automorphisms of $C(\beta \mathbb{N} \setminus \mathbb{N})$ are trivial.’*
The Calkin algebra

\[ H = \ell_2(\aleph_0) \]: an infinite-dimensional complex Hilbert space.
\[ \mathcal{B}(H) \]: The algebra of bounded linear operators.
\[ \mathcal{K}(H) \]: The ideal of compact operators.
\[ \mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H) \]: Quotient C*-algebra, the so-called *Calkin algebra.*

**Question (Brown–Douglas–Fillmore, 1977)**

Are all automorphisms \( \Phi \) of the Calkin algebra inner?

As usually, \( \Phi \) is *inner* if for some \( u \in \mathcal{C}(H) \) we have \( \Phi(a) = uau^* \) for all \( a \).
The Calkin algebra

$H = \ell_2(\mathbb{N}_0)$: an infinite-dimensional complex Hilbert space.
$\mathcal{B}(H)$: The algebra of bounded linear operators.
$\mathcal{K}(H)$: The ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$: Quotient C*-algebra, the so-called Calkin algebra.

Question (Brown–Douglas–Fillmore, 1977)
Are all automorphisms $\Phi$ of the Calkin algebra inner?
The Calkin algebra

\( H = \ell_2(\mathbb{N}_0) \): an infinite-dimensional complex Hilbert space.
\( \mathcal{B}(H) \): The algebra of bounded linear operators.
\( \mathcal{K}(H) \): The ideal of compact operators.
\( \mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H) \): Quotient C*-algebra, the so-called Calkin algebra.

**Question (Brown–Douglas–Fillmore, 1977)**

Are all automorphisms \( \Phi \) of the Calkin algebra inner?

As usually, \( \Phi \) is *inner* if for some \( u \in \mathcal{C}(H) \) we have

\[
\Phi(a) = uau^*
\]

for all \( a \).
Theorem (Phillips–Weaver, 2006)

$CH$ implies $\mathcal{C}(H)$ has $2^c$ automorphisms, (and only $c$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $d = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

TA implies all automorphisms of $\mathcal{C}(H)$ are inner.

Question

What can be said in the case when $H$ is nonseparable?

Answer

All kinds of things but not enough.
Theorem (Phillips–Weaver, 2006)

$CH$ implies $C(H)$ has $2^c$ automorphisms, (and only $c$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $C(H)$ has $2^{\aleph_1}$ automorphisms.
Theorem (Phillips–Weaver, 2006)

$CH$ implies $C(H)$ has $2^\mathfrak{c}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $C(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

$TA$ implies all automorphisms of $C(H)$ are inner.
Theorem (Phillips–Weaver, 2006)

$CH$ implies $C(H)$ has $2^c$ automorphisms, (and only $c$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{o} = \aleph_1$ then $C(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

$TA$ implies all automorphisms of $C(H)$ are inner.

Question

What can be said in the case when $H$ is nonseparable?
Theorem (Phillips–Weaver, 2006)
*CH implies $\mathcal{C}(H)$ has $2^c$ automorphisms, (and only $c$ inner automorphisms).*

Proposition (Farah, Geschke 2007)
*If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.*

Theorem (Farah, 2007)
*TA implies all automorphisms of $\mathcal{C}(H)$ are inner.*

Question
*What can be said in the case when $H$ is nonseparable?*

Answer
*All kinds of things*
Theorem (Phillips–Weaver, 2006)

$CH$ implies $\mathcal{C}(H)$ has $2^c$ automorphisms, (and only $c$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

$TA$ implies all automorphisms of $\mathcal{C}(H)$ are inner.

Question

What can be said in the case when $H$ is nonseparable?

Answer

All kinds of things but not enough.