All automorphisms of all Calkin algebras
Part I: Polish Aronzajn trees

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Aronszajn trees

Definition
An $\omega_1$-tree
Aronszajn trees

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An $\omega_1$-tree is a sequence $X_\xi$, $\xi < \omega_1$ of countable sets and commuting surjections for $\eta < \xi$

$$\pi_{\xi \eta} : X_\xi \to X_\eta.$$
Aronszajn trees

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An $\omega_1$-tree is *Aronszajn* if

$$\lim_{\xi<\omega_1} X_\xi = \emptyset.$$
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An $\omega_1$-tree is Aronszajn if

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An $\omega_1$-tree is special if there exist antichains $A_n$, for $n < \omega$, such that

$$\bigcup_{\xi < \omega_1} X_\xi = \bigcup_{n < \omega_1} A_n.$$ 
Definition

A *Polish $\omega_1$-tree* is a sequence $(X_\xi, d_\xi)$, $\xi < \omega_1$ of Polish spaces and commuting surjective contractions for $\eta < \xi$

$$\pi_{\xi\eta}: X_\xi \to X_\eta.$$
Definition

A *Polish $\omega_1$-tree* is a sequence $(X_\xi, d_\xi)$, $\xi < \omega_1$ of Polish spaces and commuting surjective contractions for $\eta < \xi$

$$\pi_{\xi \eta}: X_\xi \to X_\eta.$$  

A Polish $\omega_1$-tree is a *Polish Aronszajn tree* (PA-tree) if

$$\lim_{\xi < \omega_1} X_\xi = \emptyset.$$
Special PA-trees

Problem

What is the right definition of a ‘special PA-tree?’
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On a $\text{P}\omega_1$-tree $(X_\xi, d_\xi)$, $\xi < \omega_1$ define a pseudo-metric

$$d(x, y) = d_\eta(\pi_{\xi\eta}(x), y)$$

if $x \in X_\xi$, $y \in X_\eta$ and $\xi \geq \eta$. 

Some $A \subseteq \bigcup \xi X_\xi$ is an $\epsilon$-antichain if $d(x, y) > \epsilon$ for $x \neq y$ in $A$. 

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Definition
A $P_{\omega_1}$-tree is $\varepsilon$-special if there are $\varepsilon$-antichains $A_n$, for $n \in \mathbb{N}$, such that $(\bigcup_n A_n) \cap X_\xi$ is dense in $X_\xi$ for each $\xi < \omega_1$. 
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Lemma
If a $PA$-tree is $\varepsilon$-special then it has no cofinal branches.
Fix a PA-tree $T$:

$$X_\xi, \text{ for } \xi < \omega_1 \text{ and } \pi_{\xi\eta} : X_\xi \to X_\eta \text{ for } \omega_1 > \xi \geq \eta.$$ 

Fix $\varepsilon > 0$. 
Fix a PA-tree $T$:

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Fix $\varepsilon > 0$. Some $B = \{x_\xi : x < \omega_1\}$ is an $\varepsilon$-branch of $T$ if
MA and $\varepsilon$-special trees

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Fix $\varepsilon > 0$.
Some $B = \{x_\xi : x < \omega_1\}$ is an $\varepsilon$-branch of $T$ if

1. $x_\xi \in X_\xi$ for all $\xi$, and
2. $d(x_\eta, x_\xi) < \varepsilon$ for all $\xi > \eta$. 


Lemma

A $P_{\omega_1}$-tree $T$ either has an $\varepsilon$-branch or a ccc forcing adds an $\varepsilon/2$-antichain.
The key lemma

**Lemma**

A $P_{\omega_1}$-tree $T$ either has an $\varepsilon$-branch or a ccc forcing adds an $\varepsilon/2$-antichain.

**Pf.** Fix a countable dense $Z_\xi \subseteq X_\xi$ for each $\xi$. Let

$$\mathbb{P} = \{ p \in \bigcup_{\xi} Z_\xi : p \text{ is a finite } \varepsilon/2\text{-antichain} \}$$

ordered by $p \leq q$ iff $p \supseteq q$. 
Lemma
A $P_{\omega_1}$-tree $T$ either has an $\varepsilon$-branch or a ccc forcing adds an $\varepsilon/2$-antichain.

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ordered by $p \leq q$ iff $p \supseteq q$.
We prove that is $T$ has no $\varepsilon$-branches then $P$ is powerfully ccc, i.e., $P^{<\aleph_0}$ is ccc.
Fix $p_\xi$, for $\xi < \omega_1$, in $\mathbb{P}$. We may assume

$$p_\xi = p \cup r_\xi$$

and that for some $n$

$$r_\xi = \{x_\xi^0, \ldots x_\xi^{n-1}\}.$$
Proving $\mathbb{P}$ is (powerfully) ccc

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Fix a uniform ultrafilter $\mathcal{U}$ on $\omega_1$. 
For each $\xi$ fix $A_\xi \in \mathcal{U}$ such that for some $s_\xi : n^2 \to 2$

all $\eta \in A_\xi$ and all $(i, j) \in n^2$ we have

$$d(x^i_\xi, x^j_\eta) < \frac{\varepsilon}{2} \text{ iff } s_\xi(i, j) = 0.$$
Still proving $\mathbb{P}$ is (powerfully) ccc

For each $\xi$ fix $A_\xi \in \mathcal{U}$ such that for some $s_\xi : n^2 \to 2$ all $\eta \in A_\xi$ and all $(i, j) \in n^2$ we have

$$d(x^i_\xi, x^j_\eta) < \frac{\varepsilon}{2} \text{ iff } s_\xi(i, j) = 0.$$

Fix $A \in \mathcal{U}$ and $s$ such that for all $\xi \in A$ we have $s_\xi = s$. 
Case 1: \( s(i, j) = 0 \) for some \( i \) and \( j \).
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For \( \xi < \eta \) in \( A \) we have (using any \( \zeta \in A_\xi \cap A_\eta \))

\[
d(x_\xi^i, x_\eta^i) \leq d(\xi^i, \eta^j) + d(\eta^j, \xi^i) < \varepsilon
\]

therefore \( \{x_\xi^i : \xi \in A\} \) defines an \( \varepsilon \)-branch.
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Case 2: $s(i, j) = 1$ for all $i, j$. Then $p_\xi$, for $\xi \in A$, is linked hence $P$ is ccc. The same proof shows that $P$ is powerfully ccc.
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Lemma

MA implies that if a $P_{\omega_1}$-tree has no $\varepsilon$-branches for $\varepsilon > 0$ then it is $\varepsilon/2$-special.
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There exists a PA-tree that has an \( \varepsilon \)-branch for all \( \varepsilon > 0 \) but no branches.
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There exists a PA-tree that has an $\varepsilon$-branch for all $\varepsilon > 0$ but no branches.
This tree is even special.
A sequence $Y_\xi$, $\xi < \omega_1$ is a subtree of a $P\omega_1$-tree $T = \langle X_\xi, \pi_{\xi \eta}\rangle$ if

1. $Y_\xi \subseteq X_\xi$ for all $\xi$ and
2. $\pi_{\xi \eta}[Y_\xi] \subseteq Y_\eta$. 

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**Lemma**

*If a P$\omega_1$-tree $T$ is such that every subtree has an $\varepsilon$-branch for every $\varepsilon > 0$, then $T$ has a branch.*
Constructing an honest branch

*Pf.* Pick a $1/2$-branch $\langle x^1_\xi : \xi < \omega_1 \rangle$. 
Constructing an honest branch

Pf. Pick a $1/2$-branch $\langle x^1_\xi : \xi < \omega_1 \rangle$.
Let $X^1_\eta \subseteq X_\eta$ be the metric closure of

$$\left\{ \pi_{\xi\eta}(x^1_\xi) : \eta < \xi < \omega_1 \right\}.$$  

Then $T^1 = \langle X^1_\eta : \eta < \omega_1 \rangle$, is a subtree of $T$. 
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Pick a $1/4$-branch $x^2_\xi$, $\xi < \omega_1$, of $T^1$. 

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Let $X^2_\eta \subseteq X^1_\eta$ be the metric closure of

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Then $T^2 = \langle X^2_\eta : \eta < \omega_1 \rangle$, is a subtree of $T$.

Pick a 1/8-branch $x^3_\xi$, $\xi < \omega_1$, of $T^1$.
Let $X^3_\eta \subseteq X^2_\eta$ be the metric closure of

$$\{ \pi_{\xi \eta}(x^3_\xi) : \eta < \xi < \omega_1 \}.$$  

Then $T^3 = \langle X^3_\eta : \eta < \omega_1 \rangle$, is a subtree of $T$. 
Constructing an honest branch

\textit{Pf.} Pick a 1/2-branch \( \langle x^1_{\xi} : \xi < \omega_1 \rangle \).
Let \( X^1_\eta \subseteq X_\eta \) be the metric closure of
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\ldots
For each $\eta$ the sequence $x_\xi^n$, for $n < \omega$, is a Cauchy sequence. Then

$$x_\xi = \lim_{n} x_\xi^n$$

defines a cofinal branch in $T$. □
Proposition

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Having an $\varepsilon$-special subtree does not guarantee $T$ has no cofinal branches...
Proposition

*MA implies that every $P_{\omega_1}$-tree $T$ either has a branch or a (cofinal) $\varepsilon$-special subtree for some $\varepsilon > 0$.*

Having an $\varepsilon$-special subtree does not guarantee $T$ has no cofinal branches.

...unless we assume something extra about $T$!
An $\omega_1$-tree $T$ is **coherent** if $T \subseteq 2^{\omega_1}$ and for all $s$ and $t$ in $T$ the set
\[ \{ \xi < \omega_1 : s(\xi) \neq t(\xi) \} \]
is finite.

(Discrete) coherent trees
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**Lemma**

If a coherent $\omega_1$-tree has a cofinal branch, then each one of its cofinal subtrees has a cofinal branch.
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Coherent Polish $\omega_1$-trees

A Polish tree $T = \langle X_\xi, \pi_\xi, \eta < \xi < \omega_1 \rangle$ is coherent
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$$\inf_{y'} d_\xi(x, y') < \varepsilon$$
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$$\{ \zeta < \xi : y'((\zeta)) \neq y(\zeta) \} \subseteq F.$$
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Lemma

If a coherent $P\omega_1$-tree has a cofinal branch, then each one of its cofinal subtrees has a cofinal branch.

In particular, it cannot have a cofinal special subtree.
Polish coherent families

Let $\Lambda$ be a directed set and $X_\xi$, for $\xi \in \Lambda$ a family of Polish spaces.

Then $\langle X_\xi, \pi_{\xi\eta} \rangle$ is a Polish coherent family.

It is nontrivial if $\lim_{\xi \leftarrow \infty} X_\xi = \emptyset$.

Proposition
Assume PFA. If a Polish coherent family is nontrivial, then for some $\varepsilon > 0$ it has an $\varepsilon$-special $\omega_1$-subtree.
Polish coherent families

Let \( \Lambda \) be a directed set and \( X_\xi \), for \( \xi \in \Lambda \) a family of Polish spaces. For \( \eta < \xi \) in \( \Lambda \) we have a contractive surjection

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\pi_{\xi\eta} : X_\xi \to X_\eta.
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Polish coherent families

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Then $\langle X_\xi, \pi_{\xi\eta} \rangle$ is a Polish coherent family. It is nontrivial if

$$\lim_{\xi} \left\langle X_\xi \right\rangle = \emptyset.$$

**Proposition**

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