Definability of grounds in ZF

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Conventions

Throughout this talk,

- A forcing means a set-forcing.
- An inner model of ZF(C) means a transitive model M of ZF(C) such that $\mathrm{Ord} \subseteq M$ and the pair (M, V) satisfies the replacement scheme.
  - Every definable model is an inner model.
  - But inner models are not necessary definable in V.
  - V is an inner model of its forcing extension.

Definability of ground models

Theorem (Laver, Woodin)

In every forcing extension $V[G]$ of V, the ground model V is definable in $V[G]$ with some parameters:
There is a formula $\varphi(x, y)$ and $p \in V[G]$ such that for every $x \in V[G]$,

$$x \in V \iff V[G] \models \varphi(x, p)$$

Actually all ground models are uniformly definable.

Theorem (Fuchs-Hamkins-Reitz)

There is a formula $\varphi(x, y)$ such that:

1. For every $r \in V$, $W_r = \{x : \varphi(x, r)\}$ is a transitive model of ZFC, and is a ground model of V.
2. For every inner model $M \subseteq V$ of ZFC, if M is a ground model of V, then there is r such that $M = W_r$.

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Covering and approximation

Definition (Hamkins)

Let M be an inner model of ZFC, and κ a cardinal.

1. M satisfies the κ-covering property if for every $x \subseteq M \cap \mathrm{Ord}$ with $|x| < \kappa$, there is $y \in M$ such that $x \subseteq y$ and $|y| < \kappa$.
2. M satisfies the κ-approximation property if whenever $X \subseteq M \cap \mathrm{Ord}$, if $X \cap x \in M$ for every $x \in M$ with size $< \kappa$, then $X \in M$.

(Note: these definitions make sense in ZF)

Theorem (Hamkins)

Let M, N be inner models of ZFC, and κ a cardinal. If M and N satisfy the κ-covering and the κ-approximation properties, and $\mathcal{P}(\kappa^+) \cap M = \mathcal{P}(\kappa^+) \cap M$, then $M \cap \mathcal{P}(\mathrm{Ord}) = N \cap \mathcal{P}(\mathrm{Ord})$, in particular $M = N$.

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Definable by Covering and approximation

Fact (folklore? (Mitchell?))

Let $M \subseteq V$ be an inner model of ZFC. Suppose $V = M[G]$ for some poset $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic G. Then for every $\kappa > |\mathbb{P}|$, M satisfies the κ-covering and the κ-approximation properties.

Corollary

Suppose $V = M[G]$ for some $G \subseteq \mathbb{P} \in M$. Let $\kappa > |\mathbb{P}|$ and $X = M \cap \mathcal{P}(\kappa^+)$. Then M is definable as a unique transitive model N of ZFC such that:

1. N satisfies the κ-covering and κ-approximation properties.
2. $N \cap \mathcal{P}(\kappa^+) = X$.
3. $\mathbb{P} \in N$ and $N[G] = V$.

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1. N satisfies the κ-covering and κ-approximation properties.
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3. $\mathbb{P} \in N$ and $N[G] = V$.

Ground model definability in ZF

Forcing over models of ZF is very useful to construct various models: e.g., forcing over $L(\mathbb{R})$.

Question

Suppose V is a model of ZF, and $V[G]$ is a forcing extension of V. Does V is definable in $V[G]$ with parameters?

Can we imitate known arguments?

1. In ZF, we can prove that, in $V[G]$, V satisfies the κ-covering property for some large κ.
2. However, it is unclear that V satisfies the κ-approximation property for some κ.
3. Even if V satisfies the κ-covering and the κ-approx. properties, we have no idea how to prove V is a unique model satisfying the covering and the approximation properties.

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Known result

Theorem (Gitman-Johnstone, ZF)

Suppose DC_κ holds. Then for every poset $\mathbb{P}$ with size $< \kappa$ (hence $\mathbb{P}$ is assumed to be well-orderable), V is definable in $V^{\mathbb{P}}$ with some parameters.

Their proof does not need the full AC but a weak AC.
It is still open whether V is always definable in $V[G]$ without AC.

Main result

In this talk, we show the following partial answers of this question.

Theorem (ZF)

Suppose one of the following holds:

1. There is a poset $\mathbb{P}$ which forces the Axiom of Choice, or
2. There are proper class many supercompact cardinals.

Then there is a formula $\varphi(x, y)$ such that for every forcing extension $V[G]$ of V, there is $p \in V$ with

$$x \in V \iff V[G] \vDash \varphi(x, p)$$

Hence V is definable in $V[G]$. Actually every ground model is uniformly definable.

Corollary (ZF)

1. If $V = L(X)$ for some X, then V is definable in its forcing extensions.
2. It is consistent that DC_ω (or countable choice, or other weak AC) fails, but V is definable in its forcing extension of V.

Assumptions

1. There is a poset $\mathbb{P}$ which forces the Axiom of Choice, or
2. There are proper class many supercompact cardinals.

First we prove the theorem under the assumption (1), and next prove under (2).

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When is AC forceable?

In order to prove main theorem under the assumption (1), we need a new characterization of "AC is forceable".

Theorem (Blass, ZF)

The following are equivalent:

1. There is a poset which forces AC.
2. There is a set X such that for every set Y, there is an ordinal α is a surjection $f : \alpha \times X \to Y$.

Semi-ground

Definition (ZF)

Let $M \subseteq V$ be an inner model of ZF.
For a cardinal κ, M satisfies the κ-global covering property if for every α and $f : \alpha \to \mathrm{Ord}$, there is $F \in M$ such that $\mathrm{dom}(F) = \alpha$, $f(\beta) \in F(\beta)$ and $|F(\beta)| < \kappa$ for $\beta < \alpha$.
(Note: $F(\beta)$ can be a set of ordinals, hence $|F(\beta)| < \kappa$ makes sense.)
M is a semi-ground if M satisfies the Axiom of Choice, and there is a cardinal κ such that M satisfies the κ-global covering property.

Theorem (Bukovsky, ZFC)

Let $M \subseteq V$ be an inner model of ZFC. Then M is a ground model of V if and only if M satisfies the κ-global covering property for some κ.

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Let $M \subseteq V$ be an inner model of ZFC. Then M is a ground model of V if and only if M satisfies the κ-global covering property for some κ.

Definability of semi-grounds

Proposotion (ZF)

1. If $M \subseteq V$ is a semi-ground, then M satisfies the κ-approximation property for some κ.
2. Let M and N be inner models of ZFC. Suppose M and N satisfy the κ-global covering property, and $M \cap \mathcal{P}(\kappa^+) = N \cap \mathcal{P}(\kappa^+)$. Then $M = N$.
3. Semi-grounds are uniformly definable: There is a formula $\varphi(x, y)$ such that:
   - 3.1 For every $r \in V$, $W_r = \{x : \varphi(x, r)\}$ is a transitive model of ZFC, and is a semi-ground model of V.
   - 3.2 For every $M \subseteq V$, if M is a semi-ground model of V, then there is r such that $M = W_r$.

The proofs are the same to one in the context of ZFC; AC in M is necessary, but AC in V is not.

New characterization that AC is forceable

Definition (ZF)

Let $M \subseteq V$ be an inner model of ZF. For a set X, let $M(X)$ be the minimal transitive model of ZF with $M \cup \{X\} \subseteq M(X)$:
$M(X) = \bigcup_\alpha L(M_\alpha \cup \{X\})$.

Proposotion (ZF)

The following are equivalent:

1. There is a poset which forces AC.
2. There is a semi-ground M and a set X such that $V = M(X)$.

Proof: (2) $\Rightarrow$ (1). Force $\mathrm{Coll}(\omega, X)$, adding a surjection from ω onto X. In $V^{\mathrm{Coll}(\omega,X)}$, X is well-orderable. M satisfies the AC, hence in $V[G]$, every element of $V = M(X)$ is well-orderable. Since $V^{\mathrm{Coll}(\omega,X)}$ is a forcing extension of V, every element of $V^{\mathrm{Coll}(\omega,X)}$ is well-orderable as well.

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Theorem (U., ZFC)

Ground models are downward directed: Suppose M and N are inner models of ZFC, and ground models of V. Then there is a model W of ZFC such that W is a common ground model of M and N.

Proof of (1) $\Rightarrow$ (2).

Take a poset $\mathbb{P}$ which forces AC. Take V-generic $G, H \subseteq \mathbb{P}$ which are mutually generic. Then $V[G \times H]$ is a common forcing extension of $V[G]$ and $V[H]$. Note that $V[G], V[H]$, and $V[G \times H]$ satisfy AC.

By Solovay's result, we have $V[G] \cap V[H] = V$.

$V[G]$ and $V[G]$ are ground models of $V[G \times H]$. Hence there is a model W of ZFC which is a common ground model of $V[G]$ and $V[H]$. Then

$$W \subseteq V[G] \cap V[H] = V \subseteq V[G] = W[G']$$

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Since W is a semi-ground of $V[G]$ and $W \subseteq V \subseteq V[G]$, one can check that W is a semi-ground of V. Then $V = W(X)$ for some X by the following Gregoriff's result:

Theorem (Gregoriff, ZF)

Suppose $W \subseteq V$ is an inner model of ZF, and suppose $W \subseteq V \subseteq W[G']$. Then $W[G']$ is a forcing extension of V if and only if $V = W(X)$ for some X.

Lemma (ZF)

Suppose AC is forceable. Then for every inner model $M \subseteq V$ of ZF, if M is a ground model of V, then there is a semi-ground W of V and a set X such that $M = W(X)$.

Proof.

Since V is a forcing extension of M, AC is forceable over M. Hence there is a semi-ground W of M and a set X such that $M = W(X)$. $W \subseteq M \subseteq V$, hence one can check that W is also a semi-ground of V. □

Uniform definability of ground models

Theorem (ZF)

If AC is forceable, then all ground models are uniformly definable.

Proof.

Let $\{W_r : r \in V\}$ be a uniformly definable collection of all semi-grounds.
Then all ground models are definable as: $M \subseteq V$ is a ground model $\iff$ $M = W_r(X)$ for some r and X, and there is a poset $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic G with $M[G] = V$. □

Corollary

If AC is forceable, then V is definable in its forcing extensions.

Large cardinals in ZF

Definition (Woodin, ZF)

Let κ be an uncountable cardinal.

1. κ is inaccessible if for every $x \in V_\kappa$, there is no cofinal map from x into κ.
2. κ is supercompact if for every $\alpha > \kappa$, there is $\beta \geq \alpha$, a transitive set N, and an elementary embedding $j : V_\beta \to N$ such that:
   2.1 $\mathrm{crit}(j) = \kappa$ and $\alpha < j(\kappa)$.
   2.2 ${}^{V_\alpha}N \subseteq N$.

- Every supercompact cardinal is inaccessible.
- In ZFC, κ is inaccessible $\iff$ κ is inaccessible in the usual sense, and κ is supercompact $\iff$ κ is supercompact in the usual sense.

Norm: coarse measure of sets

In ZF, we can not define the cadinalitiy of a set in the usual sense, so the covering and the approximation properties do not work well as intended. To recover it, we define a rough and coarse measure of sets which will be work in ZF:

Definition (ZF)

For a set x, the norm $||x||$ of x is the least ordinal α such that there is a surjection from V_α onto x.

- $||x|| \leq \mathrm{rank}(x)$.
- $x \subseteq y \Rightarrow ||x|| \leq ||y||$.
- $||x|| = ||y|| \not\Rightarrow$ there is a bijection $f : x \to y$.
- If $M \subseteq V$ is an inner model of ZF and $x \in M$, then $||x|| \leq ||x||^M$.

Norm-covering and approximation

Definition (ZF)

Let $M \subseteq V$ be an inner model of ZF, and α an ordinal.

1. M satisfies the α-norm covering property if for every set $x \subseteq M$ with $||x|| < \alpha$, there is $y \in M$ such that $x \subseteq y$ and $||y||^M < \alpha$.
2. M satisfies the α-norm approximation property if whenever $X \subseteq M$, if $X \cap x \in M$ for every $x \in M$ with $||x||^M < \alpha$, then $X \in M$.

Super-inaccessible cardinal

Next, we will define large cardinals which is sufficient to imply a kind of Löwenheim-Skolem theorem:

Definition (ZF)

An inaccessible cardinal κ is super-inaccessible if for every $\alpha \geq \kappa$, $\lambda < \kappa$, and $x \in V_\alpha$, there is $\beta \geq \alpha$ and $M \prec V_\beta$ such that:

1. $M \cap \kappa \in \kappa$ and $V_{M \cap \kappa} \subseteq M$.
2. $x \in M$, and ${}^{V_\lambda}(M \cap V_\alpha) \subseteq M$.
3. If N is the transitive collapse of M, then $N \in V_\kappa$ (hence $||M|| < \kappa$).

- Every supercompact cardinal is super-inaccessible.
- In ZFC, every inaccessible cardinal is super-inaccessible.

Definability of models

Lemma (ZF)

Let $M_0, M_1 \subseteq V$ be inner models of ZF, λ a cardinal, and $\kappa > \lambda$ a super-inaccessible cardinal. If M_0 and M_1 satisfy the λ-norm covering and the λ-norm approximation properties, and $M_0 \cap V_\kappa = M_1 \cap V_\kappa$, then $M_0 = M_1$.

Proof: By the induction on the rank of sets.
Suppose $\alpha > \kappa$ and $M_0 \cap V_\alpha = M_1 \cap V_\alpha$. We show that: If $x \in M_0 \cap \mathcal{P}(V_\alpha)$ with $||x||^{M_0} < \lambda$, then $x \in M_1$.
Since κ is super-inaccessible, there is $\beta > \alpha + 1$ and $N \prec V_\beta$ such that N contains all relevant objects, ${}^{V_\lambda}(N \cap V_\alpha) \subseteq N$, and $||N|| < \kappa$. We know $x \subseteq N \cap V_\alpha \cap M_0$. By the λ-norm approx. property of M_0 and M_1, we have $N \cap V_\alpha \cap M_0 \in M_0 \cap M_1$. Let N' be the transitive collapse of $N \cap V_\alpha \cap M_0 \in M_0 \cap M_1$. $||N'|| < \kappa$, and since $V_\kappa \cap M_0 = V_\kappa \cap M_0$, we have $x \in \mathcal{P}(N \cap V_\alpha \cap M_0) \cap M_0 = \mathcal{P}(N \cap V_\alpha \cap M_0) \cap M_1$.

Covering and approximation of ground models

Lemma (ZF)

Suppose κ is a super-inaccessible cardinal. Suppose $V = M[G]$ for some $G \subseteq \mathbb{P} \in M$. If $\mathbb{P} \in V_\kappa$, then M satisfies the κ-norm covering and the κ-norm approximation properties.

Theorem (ZF)

Suppose there are proper class many super-inaccessible cardinals. Then all ground models are uniformly definable; If $V = M[G]$ for some $G \subseteq \mathbb{P} \in M$, take supercompact cardinals $\lambda < \kappa$ with $\mathbb{P} \in V_\lambda$, and let $X = M \cap V_\kappa$. Then M is definable as a unique model N of ZF such that

1. N satisfies the λ-norm covering and the λ-approximation properties,
2. $N \cap V_\kappa = X$,
3. $V = N[G]$ for some $G \subseteq \mathbb{P} \in N$.

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1. N satisfies the λ-norm covering and the λ-approximation properties,
2. $N \cap V_\kappa = X$,
3. $V = N[G]$ for some $G \subseteq \mathbb{P} \in N$.

Lemma (ZF)

If κ is super-inaccessible, and $\mathbb{P} \in V_\kappa$, then κ is super-inaccessible in $V^{\mathbb{P}}$.

Corollary (ZF)

Suppose there are proper class many super-inaccessible cardinals. Then V is definable in its forcing extensions.

Remarks

Woodin proved that some fragments of AC are derived from large cardinals:

Theorem (Woodin, ZF)

1. Suppose κ is supercompact. Then $\mathrm{Coll}(\omega, < V_\kappa)$ forces the dependent choice.
2. Suppose κ is a singular cardinal, and a limit of supercompact cardinals. Then κ^+ is regular, and the non-stationary ideal over κ^+ is κ^+-complete.

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Supercompact cardinals in Woodin's theorem can be replaced by super-inaccessible cardinals.

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Lemma (ZF)

Supercompact cardinals in Woodin's theorem can be replaced by super-inaccessible cardinals.

Conjecture (Woodin, ZF)

The Axiom of Choice Conjecture is: Suppose there are many supercompact cardinals. Then AC is forceable.

- It is known that The Axiom of Choice Conjecture is true if $V = L(\mathcal{P}(\mathrm{Ord}))$ and Woodin's HOD-Conjecture is provable from ZFC.
- The Axiom of Choice Conjecture is equivalent to:

  Suppose there are many supercompact cardinals. Then V has a semi-ground W such that $V = W(X)$ for some X.
- If the Axiom of Choice Conjecture is true, then "there are proper class many supercompact cardinals" $\Rightarrow$ "AC is forceable", hence the second main theorem is immediate from the first.

Thank you for your attention!

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