Tutorial: Ramsey theory in Forcing - Day 3

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Day 3: Forcing in Ramsey Theory

Forcing in Ramsey Theory on Trees
and Applications to Relational Structures

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5. Finite Ramsey degrees for the universal triangle-free graph.
6. Extensions to structures of measurable cardinality.

Strong Trees

A tree $T \subseteq 2^{<\omega}$ is a **strong subtree** if there is an infinite set $L \subseteq \omega$ such that for each node $t \in T$, t splits iff $\mathrm{lh}(t) \in L$.

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Thm. (Halpern-Läuchli) Given any strong trees $T_i \subseteq 2^{<\omega}$, $i < d$, $L \in [\omega]^\omega$ the levels of splitting nodes in each T_i, and a coloring

$$c : \bigcup_{l \in L} \prod_{i<d} T_i(l) \to 2,$$

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there are strong subtrees $S_i \subseteq T_i$ and an $L' \in [L]^\omega$ which is the set of the splitting levels in each S_i such that c is monochromatic on

$$\bigcup_{l' \in L'} \prod_{i<d} S_i(l').$$

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The following forcing notion $\mathbb{P}$ will add κ many generic paths $\dot{b}_{i,\alpha}$, $\alpha < \kappa$, through each T_i, $i < d$.

$$p \in \mathbb{P} \text{ iff is a partial function } p \text{ with } \mathrm{dom}(p) = d \times \vec{\delta}_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(l_p)$, for each $i < d$, where $l_p \in \omega$.

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$q \leq p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$; or else
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Remark. This is essentially Cohen forcing but on the trees.

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4. $c(p_{\vec{\alpha}}(i, \alpha_i) : i < d) = \varepsilon_{\vec{\alpha}}$.

Claim. There are $K_0 < \cdots < K_d$ infinite subsets of κ such that the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

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This follows from a judicious coloring with ω many colors and an application of the Erdős-Rado Theorem.

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For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, letting $\vec{\alpha} = \iota_e(\vec{\theta}\,)$, $\vec{\beta} = \iota_o(\vec{\theta}\,)$, and $k_{\vec{\alpha}} := |\vec{\delta}_{\vec{\alpha}}|$, let

$$\begin{aligned} f(\iota, \vec{\theta}) = {} & \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}},, \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and } \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \\ & \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle, \\ & \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{1}$$

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Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering. By the Erdős-Rado Theorem, there is $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

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Now use conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ to help build strong subtrees S_i extending the node t_i^*, $i < d$, which have the same splitting levels and all have the same c-color ε_* on each product of level sets.

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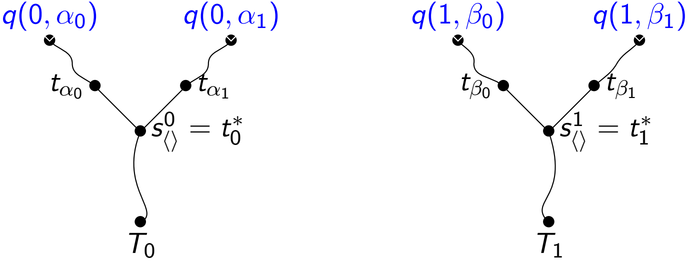

Take $q \le p_{(\alpha_0,\beta_0)}$, $p_{(\alpha_0,\beta_1)}$, $p_{(\alpha_1,\beta_0)}$, $p_{(\alpha_1,\beta_1)}$ with each $q(0,\alpha_i) \sqsupseteq t_{\alpha_i}$, and $q(1,\beta_j) \sqsupseteq t_{\beta_j}$, and deciding color ε_* for all pairs of nodes of the form $\{q(0,\alpha_i), q(1,\beta_j) : i,j < 2\}$.

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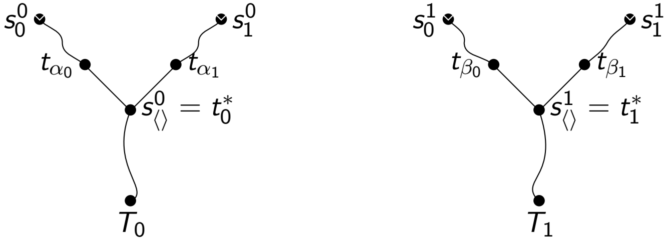

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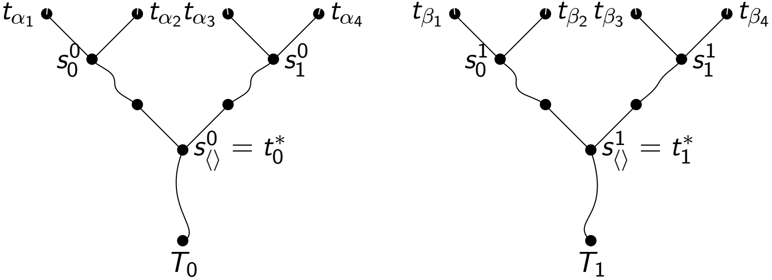

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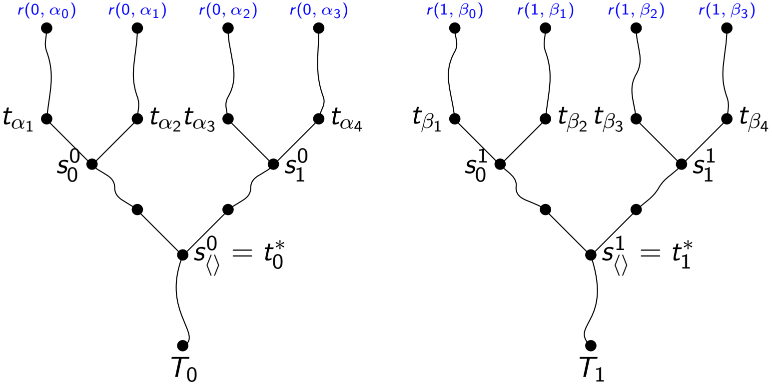

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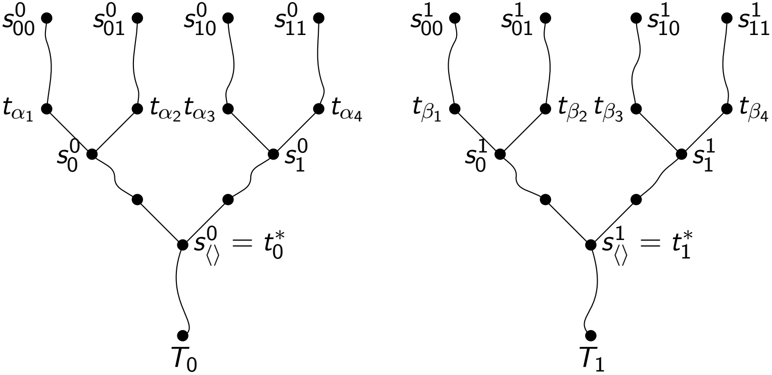

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Milliken's topological Ramsey space of strong trees ($d = 1$)

The **Milliken space** is the triple $(\mathcal{M}, \leq, r)$, where

1. $\mathcal{M}$ consists of all strong subtrees $T \subseteq 2^{<\omega}$,
2. $S \leq T$ iff S is a subtree of T,
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$\mathcal{AR}$ is the collection of all finite strong subtrees.

On board - show pictures of $r_{k+1}[k, T]$ for various $k < \omega$.

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Notice that the proof of **Axiom A.4** is really an application of the Halpern-Läuchli Theorem.

In the example of **A.4** for $r_2[1, T]$, we are applying the Halpern-Läuchli Theorem for $d = 2$ trees, one tree above each of the immediate successors of the maximal nodes in $r_1(T)$.

Exercize. Use the Halpern-Läuchli Theorem to prove **Axiom A.4** for the Milliken space.

Milliken's Theorem

For $T \in \mathcal{M}$, let $\mathcal{AR}_k|T$ denote $\{r_k(S) : S \leq T\}$.

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Milliken's Theorem was one of the key ingredients of Sauer's proof that the Rado graph has finite Ramsey degrees.

The Rado graph

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We let $\mathbf{R}$ denote the Rado graph.

Ramsey degrees

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However, for finite colorings of graphs with more than one vertex, it is not always possible to cut down to one color in a copy of the full Rado graph.

Def. The **Ramsey degree** of a finite graph G is the smallest number t_G such that for each coloring of all copies of G inside $\mathbf{R}$, there is a subgraph $\mathbf{R}'$, also a Rado graph, such that all copies of G in $\mathbf{R}'$ have no more than t_G colors.

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3. Apply Milliken's Theorem to these 'strong tree envelopes' of the trees coding G.

Coding graphs as subtrees of $2^{<\omega}$

Let G be a graph.

Enumerate the vertices of G in any order as $\langle v_n : n < \omega \rangle$.

The n-th **distinguished node** t_n codes v_n.

For all $i < n$,

$$t_n(\mathrm{lh}(t_i)) = 1 \Leftrightarrow v_n \ E \ v_i.$$

Draw

A graph.
A tree coding this graph.
A diagonal tree coding this graph.
Two types of diagonal trees coding edges.
Their strong tree envelopes.
How the Milliken Theorem is used to prove the finite Ramsey degrees for the Rado graph.

The Universal Triangle-Free Graph

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1. Construction of a new almost Ramsey space of 'strong triangle-free trees'.
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3. A new notion of subtree envelope.

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Any two countable universal homogeneous triangle-free graphs are isomorphic. Let $\mathbf{H}_3$ denote it.

Previous Results

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$$\forall G \leq H \text{ in } \mathcal{K}_3 \; \exists K \in \mathcal{K}_3 \;\; K \to (H)^G_2.$$

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Thm. (Sauer) For any finite coloring of the edges in $\mathbf{H}_3$, there is an $\mathbf{H} \in \binom{\mathbf{H}_3}{\mathbf{H}_3}$ such that the edges in $\mathbf{H}$ take on no more than two colors.

Does $\mathbf{H}_3$ have finite Ramsey degrees?

Question. Given any finite triangle free graph G, is there a number $n_{\mathrm{G}} < \omega$ such that for any coloring c of all copies of G in $\mathbf{H}_3$ into finitely many colors, there is a subgraph $\mathbf{H}$ of $\mathbf{H}_3$ isomorphic to $\mathbf{H}_3$ in which the coloring takes on no more than n_{G} colors?

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Property (A_3).

(i) $\mathbf{H}$ is triangle-free.

(ii) If A and B are disjoint finite sets of vertices of $\mathbf{H}$, and $\mathbf{H}|A$ (the graph $\mathbf{H}$ restricted to the vertices in A) does not have any edges, then there is another vertex in $\mathbf{H}$ which is connected to every member of A and to no member of B.

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Property (A_3) provides a simple way to construct $\mathbf{H}_3$. We use a slight modification in our constructions.

What codes a triangle?

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The vertices $\{v_i, v_j, v_k\}$ form a triangle if and only if the distinguished nodes t_i, t_j, t_k satisfy $t_k(|t_j|) = t_k(|t_i|) = t_j(|t_i|) = 1$.

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Here $|t|$ is denoting the length of the node t.

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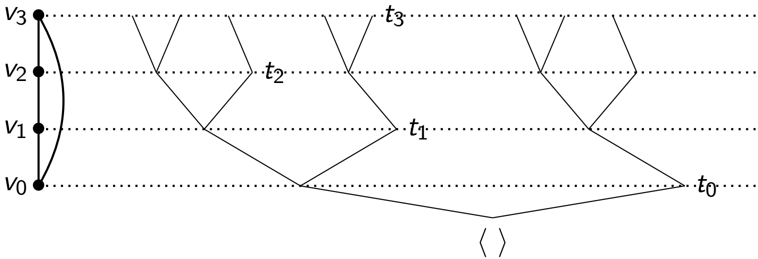
Triangle-Free Extension Criterion: A node t at the level of the n-th distinguished node t_n extends right if and only if t and t_n have no parallel 1's.

Every node always extends left.

Building a strong triangle-free $\mathbb{T}$ to code $\mathbf{H}_3$



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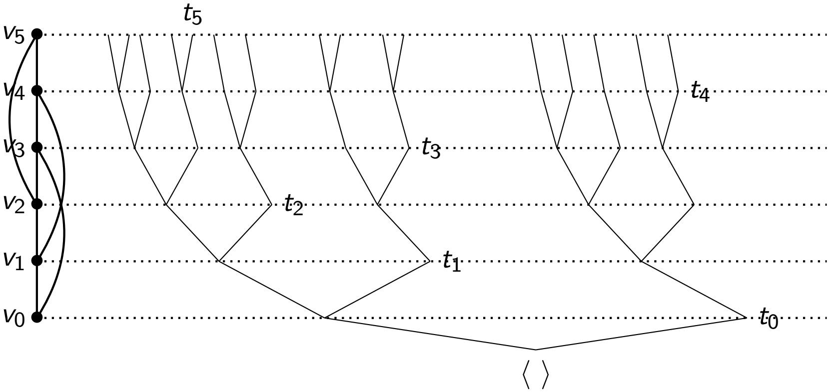

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A node in a strong triangle-free coding tree is a **critical node** if it is either a distinguished node or a splitting node.

The almost Ramsey space $\mathcal{T}(\mathbb{T})$ of strong coding trees

Let $\mathbb{T}$ be a skew strong triangle-free tree densely coding $\mathbf{H}_3$.

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$r_k(T)$ is the first k levels of T, where the levels are determined by the critical nodes.

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5. After we've finished homogenizing $\mathbb{S}$ for the finitely many triangle graphs G_i, thin $\mathbb{S}$ to a diagonal subtree $\mathbb{D}$ coding $\mathbf{H}_3$ and a set of auxiliary witnessing nodes W from $\mathbb{S}$ so that for each tree coding a tree in our collection of types there are nodes in W available to get an envelope in $\mathbb{S}$.

A.4 and stronger statements are proved by forcing

Thm. (D.) Let $T \in \mathcal{T}$, $k < \omega$, $c : r_{k+1}[r_k(T), T] \to 2$ be given. Then there is an $S \in [r_k(T), T]$ such that c is constant on $r_{k+1}[r_k(T), S]$.

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1. If we don't use skew trees, we cannot obtain **A.4** for $r_1[0, T]$. So we had to use skew trees in the forcing.
2. How to define a forcing for skew trees with distinguished nodes which is transitive, does not add triangles, but has properties similar to Cohen forcing in order to extend to homogeneous level sets?

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For each for $i \leq d$, let $T_i = \{t \in T : t \supseteq s_i\}$.

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$$p \in \mathbb{P} \text{ iff is a function of the form } p : \{d\} \cup (d \times \vec{\delta}_p) \to T(l_p),$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, satisfying

(i) $p(d)$ is the critical node of $T_d(l_p)$.

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(l_p)$.

Def. $\{t_i : i \leq d\}$ satisfies $(*)$ over A iff each $t_i \supseteq s_i$, t_d is a critical node, and

1. If s_d is extended to a distinguished node, then $A \cup \{t_i : i \leq d\}$ satisfies the Parallel 1's Criterion; and
2. If s_d is extended to a splitting node, then all parallel 1's in $A \cup \{t_i : i \leq d\}$ is witnessed by a distinguished node in A.

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$q \leq p$ if and only if either

1. $l_q = l_p$ and $q \supseteq p$; or else
2. $l_q > l_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, $q(d) \supset p(d)$, and
   - (a) For each $\delta \in \vec{\delta}_p$ and $i < d$, $q(i, \delta) \supset p(i, \delta)$,
   - (b) For each increasing sequence $(\alpha_0, \dots, \alpha_d) \in (\vec{\delta}_p)^d$ which $\{p(i, \alpha_i) : i < d\} \cup \{p(d)\}$ satisfying $(*)$ over A, then $\{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$ also satisfies $(*)$ over A.

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For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

1. $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$,
2. $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\}$ satisfies $(*)$ over A.
3. $p_{\vec{\alpha}} \Vdash$ "There is an $\varepsilon \in 2$ such that $c(\dot{b}_{\vec{\alpha}}(l)) = \varepsilon$ for $\dot{\mathcal{U}}$ many l",
4. $p_{\vec{\alpha}}$ decides a value for ε, call it $\varepsilon_{\vec{\alpha}}$.
5. $\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\} \cup \{p(d)\}$ takes value $\varepsilon_{\vec{\alpha}}$.

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For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, letting $\vec{\alpha}$ denote $\iota_e(\vec{\theta}\,)$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta}\,)$, let

$$\begin{aligned} f(\iota, \vec{\theta}) = & \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}},, \langle \langle i, j \rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and}\ \vec{\delta}_{\vec{\alpha}}(j) = \alpha_i \rangle, \\ & \langle \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i < d \rangle, \langle \langle p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) : j < k_{\vec{\beta}} \rangle : i < d \rangle, \\ & \langle \langle j, k \rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle \rangle. \end{aligned} \tag{2}$$

Let $f(\vec{\theta})$ be the sequence $\langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I} \rangle$, where $\mathcal{I}$ is given some fixed ordering.

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By the Erdős-Rado Theorem, there is $K \in [\kappa]^{\aleph_1}$ homogeneous for f.

Claim. There are $K_0 < \cdots < K_{d-1}$ infinite such that the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

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Then extend these t_i^*, $i \leq d$, using $K_0 < \cdots < K_{d-1}$ to build a member $S \in [r_k(T), T]$ such that c is constant on $r_{k+1}[r_k(T), S]$.

We conclude this tutorial with some Ramsey results for measurable cardinals.

Extension of the Halpern-Läuchli Theorem to a measurable

Thm. (Shelah) Suppose V is a model of ZFC with a measurable cardinal κ such that for λ large enough, after forcing with $\mathrm{Add}(\kappa, \lambda)$, κ remains measurable.

Then for each $1 \leq m < \omega$ and any coloring of the m-sized level sets of the tree $2^{<\kappa}$ into less than κ colors, there is a strong subtree $T \subseteq 2^{<\kappa}$ on which the coloring takes only finitely many colors.

In fact, the color depends exactly on the type of the induced subtree below the m-many nodes.

Thm. (Džamonja/Larson/Mitchell) Suppose V is a model of ZFC with a measurable cardinal κ such that for λ large enough, after forcing with $\mathrm{Add}(\kappa, \lambda)$, κ remains measurable.

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This is applied in two papers of Džamonja, Larson and Mitchell to prove that the Rado graph on κ many vertices has finite Ramsey degrees, and that colorings of the m-sized subsets of the κ-rationals has finite Ramsey degrees.

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Let $T \subseteq {}^{<\kappa}\kappa$ be regular. A tree $S \subseteq T$ is a **strong subtree** of T if S is regular and there is some $A \subseteq \kappa$ cofinal in κ such that for each $s \in S$,

1. s splits iff t has length $\zeta \in A$, and
2. For each $\zeta \in A$ and $s \in S(\zeta)$, s is maximally branching in T.

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$$c : \bigcup_{\zeta<\kappa} \prod_{i<\delta} T_i(\zeta) \to \sigma,$$

there exists a sequence of trees $\langle S_i : i < \delta \rangle$ and $A \in [\kappa]^\kappa$ such that

1. each S_i is a strong subtree of T_i as witnessed by $A \subseteq \kappa$, and
2. there is some $\sigma' < \sigma$ such that c has color σ' on $\bigcup_{\zeta \in A} \prod_{i<\delta} S_i(\zeta)$.

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Thm. (D./Hathaway) Let $d \geq 1$ be any finite integer and suppose that κ is a $\kappa + d$-strong cardinal in a model V of ZFC satisfying GCH. Then there is a forcing extension in which κ remains measurable and $\mathrm{HL}(d, \sigma, \kappa)$ holds, for all $\sigma < \kappa$.

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Open Problem. Find the exact consistency strength of $\mathrm{HL}(d, \sigma, \kappa)$ for κ a measurable cardinal.

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2. Forcing can be used to prove new Ramsey theorems in ZFC.
3. The interaction between these two fields enriches both.
4. There are many related open problems in both directions desiring your work.

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