Towers and Ultrafilters (joint work J. Brendle and B. Farkas) the various and rules of inference are al to CEFAUK athematical que con athan Verner vhice



The views presented here are my own and do not necessarily reflect those of my coauthors.



Definition A tower is a descending sequence in $([\omega]^{\omega}, \subseteq^*)$ with no lower bound.

Observation Assume CH. Then every P-point is generated by a tower.

Question

- Can the assumption of being a P-point be weakened/dropped?
- Can CH be weakened/dropped?



Observation If an ultrafilter is generated by a tower, then it is a P-point.

Question Can every ultrafilter contain a tower?

A naive attempt

Putting towers into ultrafilters



A naive attempt

- Enumerate $[\omega]^{\omega}$ as $\{X_{\alpha} : \alpha < \mathfrak{c}\}$
- ▶ start with some $T_0 \in U$ and recursively construct \subseteq^* -descending $T_\alpha \in U$; at step α let $T_{\alpha+1}$ be either $T_\alpha \cap X_\alpha$ or $T_\alpha \setminus X_\alpha$, depending on which is in U
- What to do at limits?
- ► It gets even worse: It can happen, that U contains a tower but at the same time has a basis such that no infinite sequence in the basis has a pseudointersection in U. (For all we know, all ultrafilters can have such a basis!)
- Under CH it fails (Kunen, van Mill, Mills)!



Balcar, Frankiewicz, Mills: More on nowhere dense closed P-sets, Bull. Acad. Polon. Sci. Sér. Sci. Math, 28 (1980)

- ► the paper is published in an obscure journal, not available online
- need to know topology

Theorem $% \omega ^{\ast }$ It is consistent that $\omega ^{\ast }$ can be covered by nowhere dense closed P-sets.

Translation It is consistent that every ultrafilter contains a tower.

Pull out the big guns

(an alternative proof)



Observation There always is some ultrafilter, which contains a tower!

Make all ultrafilters look alike!

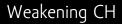
Definition (Blass) Two ultrafilters $\mathcal{V}_0, \mathcal{V}_1$ are nearly coherent if there is an ultrafilter \mathcal{U} which is RB-below both of them.

Theorem (Blass) It is consistent that every two nonprincipal ultrafilters are nearly coherent.

(under NCF)



- Observation If ${\mathcal U}$ contains a tower than all ultrafilters RB-above ${\mathcal U}$ also contain towers.
- **Theorem** If a tower \mathcal{T} of length \mathfrak{b} generates a nonmeager filter and \mathcal{V} is an ultrafilter containing \mathcal{T} , then any ultrafilter \mathcal{R} B-below contains a tower.
- Theorem (Solomon,Simon) If $\mathfrak{b} < \mathfrak{d}$ then there is a tower of length \mathfrak{b} which generates a non-meager filter.
- Note Under NCF every tower of length $\mathfrak c$ is meager! (It is not clear, whether there are towers of length $\mathfrak c.)$





- Question Assume MA. Does every P-point contain a tower?
- Question Does $\mathfrak{d} = \mathfrak{c}$ imply that every P-point contains a tower?
- Theorem $\ \ \mbox{Let}\ \omega<\lambda\leq\kappa$ be regular cardinals. Then it is consistent that
 - $\mathfrak{t} = \lambda \leq \kappa = \mathfrak{c}$; and
 - ► There is a selective ultrafilter which contains no tower.

In particular, it is consistent that $\mathfrak{d}=\mathfrak{c}$ and there is a P-point not containing a tower.

Constructing P-points without towers



- Assume $\mathfrak{t} = \mathfrak{c}$ and $\Diamond(\mathsf{S}^{\mathfrak{c}}_{\geq \omega_1})$
- ► Recursively construct the ultrafilter in c many steps
- Guarantee the following condition: there is a family $\{X_{\alpha} : \alpha < \mathfrak{c}\}$, such that if $X \in \mathcal{U}$ then $X_{\alpha} \subseteq X$ for all but boundedly many α .
- To get $\mathfrak{t} < \mathfrak{c}$, use a trick of Shelah:
 - Start with GCH and let $B = (\lambda^{<\lambda}, \supseteq)$.
 - Force $t = c = \kappa$, then force the diamond.
 - ► Finally, force with B^V.
 - This does not add reals and adds a tower of length λ thus making $\mathfrak{t} = \lambda$.
 - \blacktriangleright Finally, show that B^V does not add long towers into the ultrafilter.