Tutorial: Ramsey theory in Forcing - Day 2

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1. Rudin-Keisler and Tukey orders on ultrafilters.
2. New Ramsey-classification Theorems finding canonical equivalence relations on fronts and barriers.
3. Applications to find exact initial Rudin-Keisler and Tukey structures on ultrafilters.

Rudin-Keisler order on ultrafilters

$\mathcal{V} \leq_{RK} \mathcal{U} \Leftrightarrow$ there is a function $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{U})$,

where $h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$.

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Def. $\mathcal{X} \subseteq \mathcal{U}$ is **cofinal** in $(\mathcal{U}, \supseteq)$ iff for each $U \in \mathcal{U}$, there is an $X \in \mathcal{X}$ such that $X \subseteq U$; i.e. $\mathcal{X}$ is a filter base for $\mathcal{U}$.

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Def. $\mathcal{V}$ is **Tukey reducible** to $\mathcal{U}$ ($\mathcal{V} \leq_T \mathcal{U}$) $\Leftrightarrow$ there is a **cofinal map** from $\mathcal{U}$ into $\mathcal{V}$: $\exists f : \mathcal{U} \to \mathcal{V}$ mapping each base for $\mathcal{U}$ to a base for $\mathcal{V}$.

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Def. The **Tukey type** of $\mathcal{U}$ is the Tukey equivalence class of $\mathcal{U}$.

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Fact. $\mathcal{V} \leq_{RK} \mathcal{U}$ implies $\mathcal{V} \leq_{T} \mathcal{U}$, but not vice versa.

1. What is the structure of the Tukey types of ultrafilters?
2. How closely related are the Tukey and Rudin-Keisler orderings?
3. What is the structure of the RK classes inside a Tukey type?

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Open Problem. Is there a model of ZFC in which every ultrafilter has the maximal Tukey type?

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By **initial Tukey structure** we mean a $\leq_T$-closed collection of Tukey types of ultrafilters. These are exact structures rather than embeddings of structures. Topological Ramsey spaces are useful for finding initial Tukey structures, as we shall see today.

Ramsey Ultrafilters are Tukey-minimal

Def. An ultrafilter $\mathcal{U}$ is **Ramsey** if for each $c : [\omega]^2 \to 2$, there is a $U \in \mathcal{U}$ such that c is monochromatic on $[U]^2$.

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Def. The **Fubini product** of $\mathcal{U}$ and $\mathcal{V}$ is

$$\{X \subseteq \omega \times \omega : \{i \in \omega : \{j \in \omega : (i,j) \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

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Thm. (Todorcevic in [Raghavan/Todorcevic 12]) If $\mathcal{U}$ is Ramsey, $\mathcal{V}$ is non-principal, and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ is isomorphic to a countable Fubini iterate of $\mathcal{U}$.

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Initial Structures known so far

The following are the initial Tukey and Rudin-Keisler structures and classification of the Rudin-Keisler types within the Tukey types obtained so far.

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2. The initial Rudin-Keisler structure below $\mathcal{U}_\alpha$ is exactly the linear order $(\alpha + 1)^*$.
3. For each $\mathcal{V} \leq_T \mathcal{U}_\alpha$, the Tukey type of $\mathcal{V}$ consists precisely of the isomorphism classes of iterated Fubini products of ultrafilters from among a fixed countable collection of rapid p-points, which are represented by the canonical equivalence relations.

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Initial structures for hypercube ultrafilters

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1. The initial Tukey structure below $\mathcal{U}_k$ is the Boolean algebra $\mathcal{P}(k)$.
2. The initial Rudin-Keisler structure below $\mathcal{U}_k$ is also $\mathcal{P}(k)$.
3. If $\mathcal{V} \leq_T \mathcal{U}_k$, then the Tukey type $[\mathcal{V}]_T$ consists of all isomorphism types of Fubini iterates of certain p-points represented by canonical projection maps.

Initial Tukey and Rudin-Keisler structures differing

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3. If $\mathcal{V} \leq_T \mathcal{U}$, then the Tukey type $[\mathcal{V}]_T$ consists of all RK types of Fubini iterates of p-points essentially coded by members of $\mathcal{K}_0 \times \cdots \times \mathcal{K}_n$.

Initial Tukey and RK structures of non-p-points

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2. The initial RK structure below $\mathcal{W}_k$ is exactly a chain of length k.

Initial Tukey and RK structures of size $\mathfrak{c}$ for non-p-points

Extending the recursive construction of $\mathrm{Fin}^{\otimes k}$ to Fin^{B}, for B a uniform barrier on ω, let $\mathcal{W}_B$ denote the ultrafilter forced by $\mathcal{P}(B)/\mathrm{Fin}^{B}$.

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Thm. (D.) $\mathcal{W}_B$ is not a p-point and

1. The initial RK structure below $\mathcal{G}_B$ is a linear order of size $\mathfrak{c}$ which is isomorphic to a certain non-standard model of ω.

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1. The initial RK structure below $\mathcal{G}_B$ is a linear order of size $\mathfrak{c}$ which is isomorphic to a certain non-standard model of ω.
2. The initial Tukey structure below $\mathcal{G}_B$ contains a copy of the initial Rudin-Keisler structure below $\mathcal{G}_B$, but also contains more.

Now we give an idea of how these results were obtained.

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This proof outline can be extended to topological Ramsey spaces.

Exact Tukey and RK Structures: General Proof Outline

Let $\mathcal{R}$ be a topological Ramsey space, $\mathcal{U}$ be the filter forced by $(\mathcal{R}, \leq^*)$, and suppose $\mathcal{V} \leq_T \mathcal{U}$. Let $f : \mathcal{U} \to \mathcal{V}$ be a monotone cofinal map. Wlog assume $\mathcal{V}$ is an ultrafilter on base set ω.

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(1) Prove f is continuous in the metric topology on $\mathcal{R}$ when restricted below some member of $\mathcal{U}$.

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(5) Given $X \in \mathcal{U}$, define

$$\mathcal{F}|X = \{a \in \mathcal{F} : \exists k < \omega \, (a \leq_{\text{fin}} r_k(X))\}.$$

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(6) Define an $\mathcal{U} \restriction \mathcal{F}$ to be the filter on base set $\mathcal{F}$ generated by the set of all $\mathcal{F}|X$, $X \in \mathcal{U}$. Prove $\mathcal{U} \restriction \mathcal{F}$ is an ultrafilter.

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(9) Apply this to the function g on $\mathcal{F}$, since $g : \mathcal{F} \to \omega$ induces an equivalence relation on $\mathcal{F}$.

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(9) Apply this to the function g on $\mathcal{F}$, since $g : \mathcal{F} \rightarrow \omega$ induces an equivalence relation on $\mathcal{F}$.

(10) Decode $g(\mathcal{F} \restriction \mathcal{U})$. Often, but not always, this is isomorphic to a Fubini iterate of p-points which are determined by the products of substructures on the blocks.

Exact Tukey and RK Structures: General Proof Outline

(6) Define an $\mathcal{U} \restriction \mathcal{F}$ to be the filter on base set $\mathcal{F}$ generated by the set of all $\mathcal{F}|X$, $X \in \mathcal{U}$. Prove $\mathcal{U} \restriction \mathcal{F}$ is an ultrafilter.

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A general proof of (1) - (7) for a large class of spaces is given in [DMT].

The remainder of this day's tutorial concentrates on canonical equivalence relations on fronts and some specific examples.

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We begin at the beginning.

Erdős-Rado Theorem

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An equivalence relation E on $[\omega]^k$ is **canonical** iff there is some $I \subseteq k$ such that $\mathrm{E} = \mathrm{E}_I$, where for $a = \{a_0, \dots, a_{k-1}\},\ b = \{b_0, \dots, b_{k-1}\} \in [\omega]^k$,

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Remark. E_I can be thought of as a projection map π_I, where $\pi_I(a) = \{a_i : i \in I\}$. Then $a \ \mathrm{E}_I \ b$ iff $\pi_I(a) = \pi_I(b)$.

Exercise. The Erdős-Rado Theorem implies Ramsey's Theorem.

Fronts and Barriers on $[\omega]^\omega$

Def. $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a **front** on $[\omega]^\omega$ iff

(i) $\forall X \in [\omega]^\omega$, $\exists a \in \mathcal{F}$ such that $a \sqsubset X$; and

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Galvin's Lemma. Any front is a barrier when restricted to some small enough infinite subset of ω.

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Uniform fronts of higher rank are made recursively from those of lower rank.

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(a) *inner*, i.e. $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$, and

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Pudlak-Rödl Canonization Thm. For every front (barrier) $\mathcal{F}$ on ω and every equivalence relation E on $\mathcal{F}$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction (\mathcal{F}|M)$ is represented by an irreducible map defined on $\mathcal{F}|M$.

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Exercise. For a uniform barrier of finite rank, the Pudlák-Rödl Theorem gives back the Erdős-Rado Theorem.

Fronts on $(\mathcal{R}, \leq, r)$

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A similar theorem to Galvin's Lemma allows us to interchange fronts and barriers in general topological Ramsey spaces, so we use fronts since they are simpler.

Now we focus on canonical equivalence relations for new topological Ramsey spaces.

Canonical Equivalence Relations for Products of Fraïssé Classes

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By work of Nešetřil and Rödl (77 and 83), the Fraïssé classes of finite graphs, finite graphs omitting k-cliques, and other classes satisfy the conditions of the previous theorem.

We now go through how canonical projection maps decode the RK types of all ultrafilters Tukey reducible to the generic ultrafilter forced by the Ramsey space where blocks are from $\mathcal{K}_3$, the finite triangle-free graphs.

Board work.

We now turn an interesting class of non-p-points.

The forcing $\mathcal{P}(\omega \times \omega)/\mathrm{Fin}^{\otimes 2}$

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Thm. (Blass/D./Raghavan) $\mathcal{G}_2$ is not a p-point, is a weak p-point, satisfies $\mathcal{U} \to (\mathcal{U})^2_{k,4}$, has exactly one RK-predecessor $\pi_0(\mathcal{G}_2)$, is not Tukey maximum, and is Tukey strictly above its projected Ramsey ultrafilter $\pi_0(\mathcal{G}_2)$.

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This left open what exactly is the initial Tukey structure below $\mathcal{G}_2$.

2-Dimensional Ellentuck Space dense in $(\mathrm{Fin} \times \mathrm{Fin})^+$

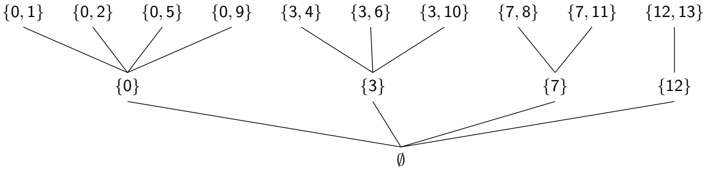


Figure: $\mathbb{W}_2 \subseteq [\omega]^2$; $\mathrm{o.t.}(\mathbb{W}_2) = \omega^2$

$X \in \mathcal{E}_2$ iff $X \subseteq \mathbb{W}_2$ tree-isomorphic to $\mathbb{W}_2$ (so $\mathrm{o.t.}(X) = \omega^2$) and respects the order of the labels on the nodes.

$Y \leq X$ iff $Y \subseteq X$.

$\mathcal{E}_2$ is a dense subset of $(\mathrm{Fin} \times \mathrm{Fin})^+$.

slide 85 and 86

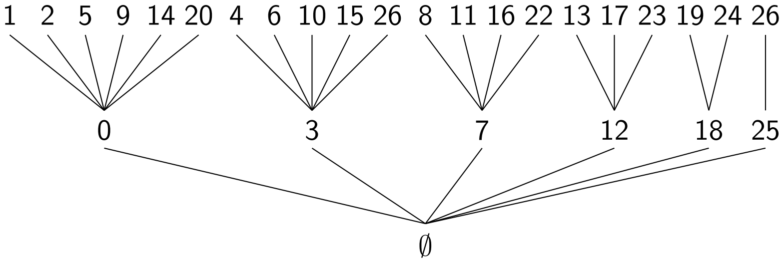
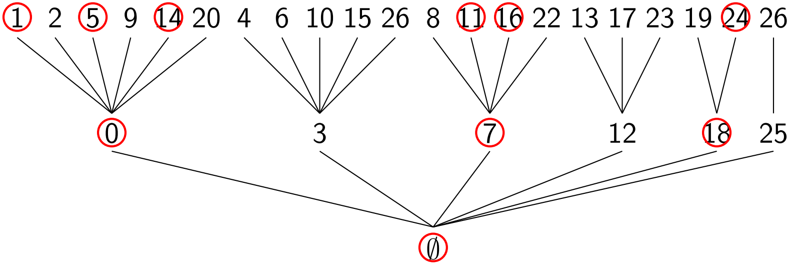


Figure: Two members X and Y of $\mathcal{E}_2$ with $Y \leq X$, and a $Z \notin \mathcal{E}_\in$.

slide 85 and 86



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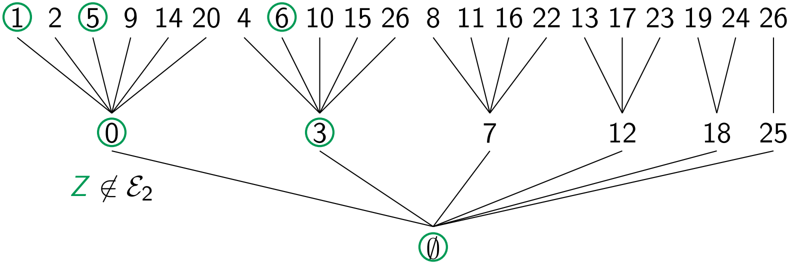

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The 2-dimensional Ellentuck spaces

Thm. (D.) $\mathcal{E}_2$ is a topological Ramsey space, and $(\mathcal{E}_2, \leq^*)$ is forcing equivalent to $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$.

The Tukey and RK structures below $\mathcal{G}_2$ are obtained using the canonical projection maps, similarly as in $\mathcal{A}_3$.

The initial Tukey structure below $\mathcal{G}_2$ consists only of $\mathcal{G}_2$ and the projected Ramsey ultrafilter below it.

Likewise the initial RK structure below $\mathcal{G}_2$ is also a chain of length 2.

$\mathcal{E}_3$, dense in $\mathcal{P}(\omega^3)/\mathrm{Fin}^{\otimes 3}$

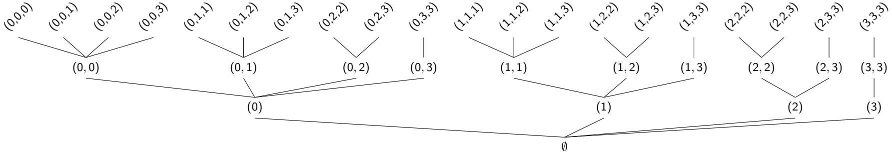

Figure: $\omega^{\not\downarrow\leq 3}$

$$\emptyset \prec (0) \prec (0,0) \prec (0,0,0) \prec (0,0,1) \prec (0,1) \prec (0,1,1) \prec (1) \prec (1,1) \prec$$

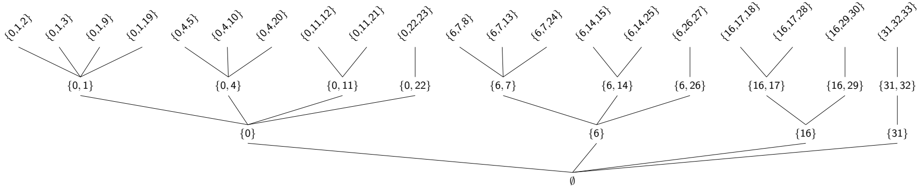

Figure: The maximum member of $\mathcal{E}_3$, $\mathbb{W}_3 \subseteq [\omega]^3$

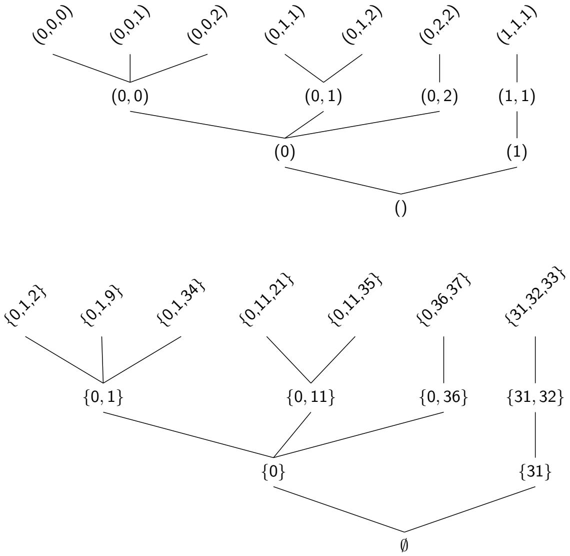

Figure: $r_7(Y)$, a typical finite approximation to a member of $\mathcal{E}_3$

The first infinite dimensional Ellentuck space

Let $\mathcal{S}$ denote the *Schreier barrier* $\{a \in [\omega]^{<\omega} : |a| = \min(a) + 1\}$.

$X \subseteq \mathcal{S}$ is in $\text{Fin}^{\mathcal{S}}$ iff for all but finitely many i, $X_i \in \text{Fin}^{\otimes i}$.

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$\mathcal{P}(\mathcal{S})/\mathrm{Fin}^{\mathcal{S}}$ forces an ultrafilter $\mathcal{G}_{\mathcal{S}}$ on base set $\mathcal{S}$.

We use the form of $\mathcal{S}$ to make our template structure of finite non-decreasing sequences of natural numbers.

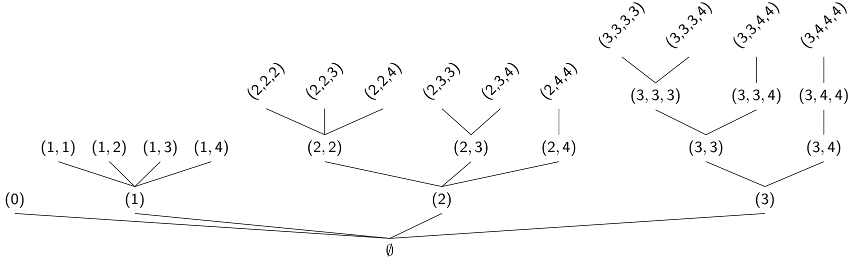
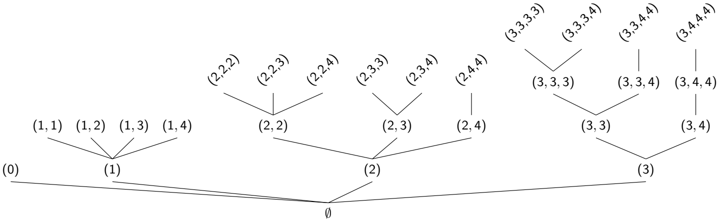


Figure: $\omega^{\not\downarrow \mathcal{S}}$

$() \prec (0) \prec (1) \prec (1,1) \prec (1,2) \prec (2) \prec (2,2) \prec (2,2,2) \prec (1,3) \prec (2,2,3) \prec (2,3) \prec (2,3,3) \prec (3) \prec (3,3) \prec (3,3,3) \prec (3,3,3,3) \prec (1,4) \prec \dots$



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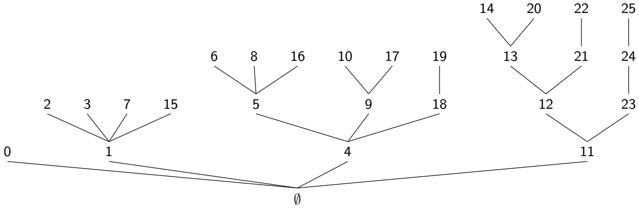

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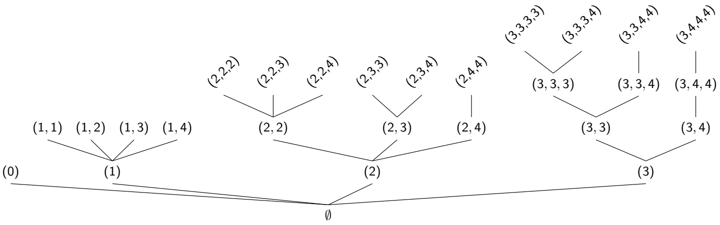

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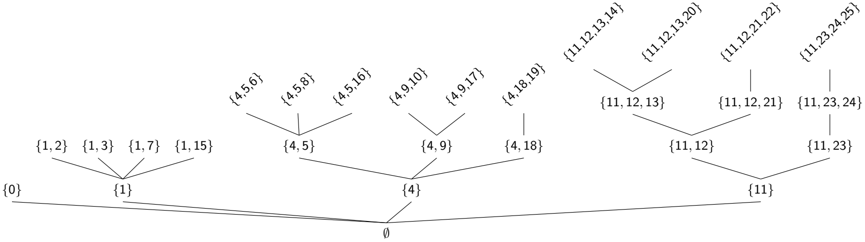

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These spaces include $\mathcal{P}(\omega^\alpha)/\mathrm{Fin}^{\otimes\alpha}$ for all $\alpha < \omega_1$.

Ramsey degrees for ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$

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3. $r(\mathcal{E}_5, 2) = 175.$
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Moreover, there is a recursive formula for finding $r(\mathcal{E}_k, 2)$ for any $k < \omega$.

On Thursday, we will look at the Halpern-Läuchli Theorem, topological Ramsey spaces of strong trees, and applications to universal relational structures.

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