# On some downwards transfer properties on Foreman-Laver model

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# Kunen's Model

## Theorem (Kunen)

*Assume there exists a huge cardinal $\kappa$ with target $\lambda$. Then there exists $W$ a forcing extension of $V$ such that $\kappa = (\aleph^1)^W$, $\lambda = (\aleph_2)^W$ and there exists an $\aleph_2$-saturated $\aleph_1$-ideal in $W$.*

- Magidor modified the construction to lessen the hypothesis on $\kappa$ from huge to almost huge.
- Magidor, Foreman and Shelah actually proved that MM implies the existence of an $\aleph_2$-saturated $\aleph_1$-ideal.
- The above model by Kunen satisfies Chang's conjecture.
- By modifying Kunen's construction, Foreman and Laver constructed a model with some nice transfer properties from $\aleph_2$ to $\aleph_1$.

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Recall:

## Definition (Chromatic number)

*Given a graph $\mathcal{G} = \langle V, E \rangle$, a function $f : V \longrightarrow \kappa$ is called a $\kappa$-**coloring** of $\mathcal{G}$ if no adjacent vertex have the same value by $f$. The **chromatic number** of $\mathcal{G}$ ($\mathrm{Chr}(\mathcal{G})$) is the smallest cardinal $\kappa$ for which there exists a $\kappa$-coloring.*

We denote:

## Definition (Transfer of the chromatic number)

*Given cardinals $\gamma < \delta$, we denote by $\mathrm{Tr}_{\mathrm{Chr}}(\delta, \gamma)$ the statement:*
*" Every graph of size and chromatic number $\delta$ has a subgraph of size and chromatic number $\gamma$."*

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# The original result

**Theorem (Foreman-Laver)**

*Assume there exists a huge cardinal $\kappa$ with target $\lambda$. Then there exists $W$ a forcing extension of $V$ such that $\kappa = (\aleph^1)^W$, $\lambda = (\aleph_2)^W$ and which satisfies $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$.*

Notice that the failure of the same transfer property is consistent:

**Theorem (Baumgartner)**

*Assume CH. Then, there exists a countably closed $\aleph_2$-c.c. poset $\mathbb{B}$ of size $\aleph_2$ which forces $\neg\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$.*

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# Construction of the model - 1

- Suppose $\kappa$ is a huge cardinal. Let $j : V \longrightarrow M$ be an elementary embedding with crit($j$)=$\kappa$, $\lambda := j(\kappa)$ and $M^\lambda \subset M$.
- the construction consists in building a two step iteration $\mathbb{P} * \dot{\mathbb{R}}$, and $V^{\mathbb{P} * \dot{\mathbb{R}}}$ will be the desired model.
- $\mathbb{P}$ collapses $\kappa$ to $\aleph_1$, while $\mathbb{R}$ collapses $\lambda$ to $\aleph_2$.
- $\mathbb{P}$ is $\kappa$-c.c., while $\mathbb{R}$ is $\lambda$-c.c and $< \kappa$-closed.

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# Constructing the model - 2

- We fix $G * H$ a $\mathbb{P} * \dot{\mathbb{R}}$ generic over $V$.
- We can find $\widehat{G} * \widehat{H}$ a $j(\mathbb{P}) * j(\dot{\mathbb{R}})$ generic over $V$ such that $j : V \longrightarrow M$ can be extended to an elementary embedding $J : V[G * H] \longrightarrow M[\widehat{G} * \widehat{H}]$.
- $\widehat{G}$ is chosen so that $V[G * H] \subset V[\widehat{G}]$.
- $V[\widehat{G}]$ is a generic extension of $V[G * H]$ by a $\kappa$-centred forcing poset (union of $\kappa$-many centred sets of conditions).
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# Sketch of the proof - 1

We show that $V[G * H] \models$ " $\mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1)$ ".

- In $V[G * H]$, let $\mathcal{G} = \langle \lambda, E \rangle$ be a graph of chromatic number $\lambda = \aleph_2$.
- We want to show that

$$V[G * H] \models \text{“} \exists \mathcal{G}' \subseteq \mathcal{G}, |\mathcal{G}'| = \mathrm{Chr}(\mathcal{G}') = \aleph_1 = \kappa \text{”}$$

- By elementarity, it is enough to show that :

$$M[\widehat{G} * \widehat{H}] \models \text{“} \exists \mathcal{G}' < J(\mathcal{G}), |\mathcal{G}'| = \mathrm{Chr}(\mathcal{G}') = J(\kappa) = \lambda = \aleph_1 \text{”}$$

- we claim that $J''\mathcal{G}$ is such a subgraph. We use the assumption $M^\lambda \subset M$ to show that indeed $J''\mathcal{G} \in M[\widehat{G} * \widehat{H}]$.

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- Suppose, towards a contradiction, that $J''\mathcal{G}$ has chromatic number $\aleph_0$ in $M[\widehat{G} * \widehat{H}]$.
- Since $J''\mathcal{G} \approx \mathcal{G}$ in $V[\widehat{G} * \widehat{H}]$, we have that $\mathcal{G}$ has countable chromatic number in $V[\widehat{G} * \widehat{H}]$.
- The forcing poset to get from $V[\widehat{G}]$ to $V[\widehat{G} * \widehat{H}]$ is $< \lambda$ close, so we can apply the following lemma:

**Lemma ($< \lambda$-closed transfer)**

*Let $D$ a $< \lambda$-closed poset. If $D$ forces a graph $\mathcal{G}$, $|\mathcal{G}| = \lambda$, to have countable chromatic number, then $\mathcal{G}$ has countable chromatic number.*

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# Sketch of the proof - 3

We also have that $V[\widehat{G}]$ is a $\kappa$-centred generic extension of $V[G * H]$, so we can apply the next lemma:

**Lemma ($\kappa$-centred transfer)**

*Let $C$ a $\kappa$-centered poset. Suppose that $C$ forces a graph $\mathcal{G}$, $|\mathcal{G}| > \kappa$, to have countable chromatic number. Then the chromatic number of $\mathcal{G}$ is $\leq \kappa$.*

We just proved that in $V[G * H]$, we have $\mathrm{Chr}(\mathcal{G}) \leq \kappa$. But this contradicts the assumption

$$V[G * H] \models \text{“}\mathrm{Chr}(\mathcal{G}) = \aleph_2 = \lambda > \kappa\text{”}$$

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- On the same Foreman-Laver model, Tall applied the same technique to obtain some results about some topological properties.
- Instead of transferring down properties from $\aleph_2$ to $\aleph_1$, his results go the opposite way, transferring properties from $\aleph_1$ to $\aleph_2$, or higher.
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## Definition

- *A collection $\mathcal{Y}$ of subsets of a topological space is* ***discrete*** *if each point in the space has a neighbourhood which meets at most one member of the collection.*
- *A discrete collection $\mathcal{Y}$ is* ***separated*** *if there exist pairwise disjoint open sets $U_Y$ for each $Y \in \mathcal{Y}$ such that $Y \subseteq U_Y$.*
- *A space is* ***$\kappa$-collectionwise Hausdorff*** *if every discrete collection of size $\leq \kappa$ consisting of points (singletons) is separated.*
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## Definition ($\kappa$-paracompactness)

- *Recall that a refinement of a cover $\mathcal{U}$ of a topological space $X$ is a cover $\mathcal{V}$ of $X$ such that for every $V \in \mathcal{V}$ there is some $U_V \in \mathcal{U}$ such that $V \subset U_V$.*
- *We say that $\mathcal{V}$ is **locally finite** if every point in the space has a neighbourhood which meets finitely many elements of $\mathcal{V}$.*
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Similarly to the proof of the transfer of the chromatic number, Tall proved the following results:

**Theorem (Tall)**

*In the model $V[G * H]$, let $X$ be a first countable topological space.*

(a) *If $X$ is weakly-$\aleph_1$-collectionwise Hausdorff, then $X$ is weakly-$\aleph_2$-collectionwise Hausdorff.*

(b) *If $X$ is locally $\aleph_2$-c.c. and $\aleph_1$-collectionwise Hausdorff, then $X$ is $\aleph_2$-collectionwise Hausdorff.*

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# Some counterexamples

**Example (Rudin-Tall)**

*Suppose there exists $S \subseteq E^{\omega_2}_{\omega_0} = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega_0\}$ which is stationary and does not reflect. Then (c) fails.*

**Example (Fleissner)**

*Suppose the existence of S like above. Then (b) fails.*

**Example (LaBergue-Landver)**

*Suppose $\square_{\omega_1}$. Then there exists a first countable $\aleph_1$-collectionwise Hausdorff space which is not weakly $\aleph_2$-collectionwise Hausdorff (hence (a) fails).*

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# Motivation

Going back to the transfer property of the chromatic number, it is easy to see that:

$$\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_2) \wedge \mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1) \Longrightarrow \mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1)$$

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# New result

**Theorem**

*Assuming the existence of a huge cardinal, then there exists $W$ a forcing extension of $V$ such that*

$$W \models \text{“} \mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1) \wedge \neg \mathrm{Tr}_{\mathrm{Chr}}(\aleph_2, \aleph_1) \text{”}$$

# Sketch of the construction

- Suppose the existence of a huge cardinal $\kappa$ with target $\lambda$.
- We can modify the poset $\mathbb{P} * \dot{\mathbb{R}}$ to create $\mathbb{P}' * \dot{\mathbb{R}}'$ such that

$$\mathbb{P}' * \dot{\mathbb{R}}' \Vdash \text{“} k = \aleph_1 \wedge \kappa^+ = \aleph_2 \wedge \lambda = \aleph_3 \wedge \mathsf{Tr}_{\mathsf{Chr}}(\aleph_3, \aleph_1) \text{”}$$

- Let $\mathbb{B}$ be the poset created by Baumgartner. By taking advantage of the small size and countable closure of $\mathbb{B}$, we can construct $\mathbb{P}''$ such that $\mathbb{P}'' * \dot{\mathbb{R}}' * \dot{\mathbb{B}}$ still forces the above.
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We can use the same argument with Tall's result (a) . Instead of $\mathbb{B}$, we use a poset which forces $\Box_{\omega_1}$. We obtain:

### Theorem

*Suppose the existence of a huge cardinal. Then there exists a forcing extension $W$ of $V$ in which every first countable weakly $\aleph_1$-collectionwise Hausdorff space is $\aleph_3$-collectionwise Hausdorff, but there exists some first countable $\aleph_1$-collectionwise Hausdorff which is not weakly $\aleph_2$-collectionwise Hausdorff in $W$.*

# Generalizing Baumgartner's poset

By changing the parameters on the construction of the poset $\mathbb{B}$ by Baumgartner, we obtain:

**Theorem**

*Let $\kappa$ be regular. Assume that $2^\kappa = \kappa^+$. Then there is a $\leq \kappa$-closed $\kappa^{++}$-c.c. poset $\mathbb{B}(\kappa)$ of size $\leq \kappa^{++}$ which adds a graph of size and chromatic number $\kappa^{++}$ such that all subgraphs of size $\leq \kappa^+$ have chromatic number $\leq \kappa$.*

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# Generalization of the new result

Using the above generalization together with the previous argument, we obtain the generalized version:

## Theorem

*Suppose* $\kappa$ *is a huge cardinal with target* $\lambda$*. Let* $\alpha$ *be an ordinal such that* $(\aleph_\alpha)^V < \kappa$ *is regular. Let also* $n + 2 < m < \omega$*. Then, there exists* $W$ *a generic extension of* $V$ *such that* $(\aleph_{\alpha+1})^W = \kappa$*,* $(\aleph_{\alpha+m})^W = \lambda$ *and*

$$W \models \text{“}\,\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\alpha+m}, \aleph_{\alpha+1}) \wedge \neg\mathrm{Tr}_{\mathrm{Chr}}(\aleph_{\alpha+n+2}, \aleph_{\alpha+n+1})\,\text{”}$$

## Question

*Is it consistent (modulo some large cardinal assumption) that*

$$\mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_1) \not\Longrightarrow \mathrm{Tr}_{\mathrm{Chr}}(\aleph_3, \aleph_2)?$$

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Thank you very much!

# References


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