The left side of Cichoń's diagram

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Joint work with Martin Goldstern and Saharon Shelah

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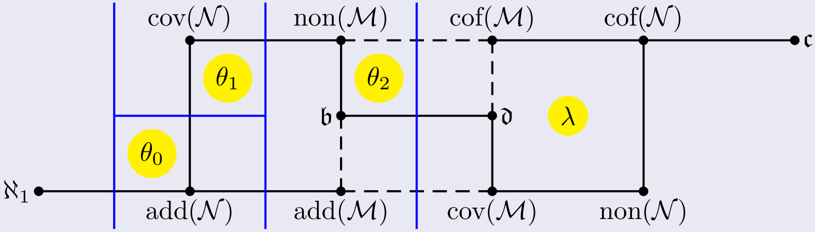
$\mathbb{E}$: Stantard Suslin σ-centered poset which adds an eventually different real (works to increase $\mathrm{non}(\mathcal{M})$).

Consistency example (1)

Theorem (From Brendle, Judah and Shelah's fsi of ccc posets techniques 1990's)

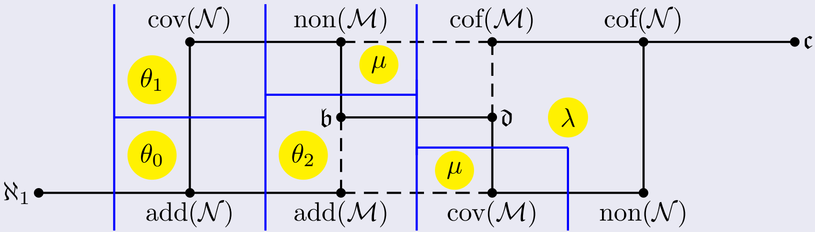
If $\theta_0 \leq \theta_1 \leq \theta_2$ are uncountable regular cardinals and $\lambda^{<\theta_2} = \lambda$, then it is consistent that



Consistency example (2)

Theorem (From Brendle, Judah and Shelah's fsi of ccc posets techniques 1990's)

If $\theta_0 \leq \theta_1 \leq \theta_2 \leq \mu$ are uncountable regular cardinals and $\lambda \geq \mu$ such that $\lambda^{<\theta_2} = \lambda$, then it is consistent that



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Theorem (Miller 1981)

$\mathbb{E}$ does not add dominating reals.

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Theorem (Pawlikowski 1992)

There are posets of the form $\mathbb{E}^N$ (as above) adding dominating reals.

Eventually different real forcing

For $\varphi : \omega \to [\omega]^{<\aleph_0}$ define the *width of* φ as

$$\mathrm{wd}(\varphi) := \min\{n^* \leq \omega : \forall i < \omega(|\varphi(i)| \leq n^*)\}$$

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Clearly, $\mathbb{E}$ is Suslin σ-centered and the generic real $e := \bigcup\{s : \exists\varphi((s, \varphi) \in G)\}$ is eventually different over the ground model.

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Fact

Note that, if there is an $m < \omega$ such that $\mathrm{wd}(\varphi_n) \leq m$ for all $n < \omega$, then its D-limit has also width $\leq m$.

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A sequence $\bar{p} = \langle p_n : n < \omega \rangle$ of conditions from $\mathbb{E}$ is *uniform* if there are $s \in \omega^{<\omega}$ and an $m < \omega$ such that each $p_n = (s, \varphi_n)$ with $\mathrm{wd}(\varphi_n) \leq m$.

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If $\bar{p} = \langle p_n : n < \omega \rangle$ is an uniform sequence in $\mathbb{E}$ then $\lim^D \bar{p}$ forces that $\{n < \omega : p_n \in \dot{G}\}$ is infinite. Moreover, there is an $\mathbb{E}$-name $\dot{D}^*$ of an ultrafilter on ω containing D such that, for every uniform sequence $\bar{p}$ in $\mathbb{E}$, $\lim^D \bar{p}$ forces

$$\{n < \omega : p_n \in \dot{G}\} \in \dot{D}^*.$$

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(ii) for $\alpha \in \mathrm{supp}(q)$, $q(\alpha)$ is some ultrafilter limit of $\langle p_{\xi_n}(\alpha) : n < \omega \rangle$.

Attempt

Claim

There are $i^* \geq m^*$ and an increasing sequence $\langle \xi_n : n < \omega \rangle$ in ω_1 such that $\langle f_{\xi_n}(i^*) : n < \omega \rangle$ is increasing.

<u>Point</u>: Find a $q \in \mathbb{P}$ that forces "$\dot{A} := \{n < \omega : p_{\xi_n} \in \dot{G}\}$ infinite" to yield a contradiction and conclude that F is preserved unbounded.

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Moreover, if this ultrafilter is $\dot{D}_\alpha$, we need that

$$\Vdash_{\mathbb{P}_\alpha} \dot{D}_\alpha \cap V^{\mathbb{P}'_\alpha} \in V^{\mathbb{P}'_\alpha}$$

to guarantee $\Vdash_{\mathbb{P}_\alpha} q(\alpha) \in \dot{\mathbb{Q}}_\alpha$.

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Then, there is some $\mathbb{P}'_\alpha \lessdot \mathbb{P}_\alpha$ of size $\leq \max\{\chi, \mu\}^{\aleph_0}$ so that $\dot{F}_\alpha$ is a $\mathbb{P}'_\alpha$-name and there is a $\mathbb{P}'_\alpha$-name for $\dot{D}^\varepsilon_\alpha \cap V^{\mathbb{P}'_\alpha}$.

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So $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}'_\alpha$-name for $\mathbb{E}^{V^{\mathbb{P}'_\alpha}}$.

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Limit step: $\mathbb{P}_\alpha$ is just the direct limit, but D^ε_α should be constructed so that, for any uniform sequence $\bar{p}$ from $\mathbb{P}_\alpha$ whose limit can be calculated by the ε-th chain, its limit forces

$$\{n<\omega : p_n \in \dot{G}_\alpha\} \in \dot{D}^\varepsilon_\alpha.$$

Construction of chains of ultrafilters

Fact

A uniform sequence $\bar{p}$ defines a countable partial function $c^{\bar{p}} = (c_0^{\bar{p}}, c_1^{\bar{p}})$ from δ into $\omega^{<\omega} \times \omega$ with $\mathrm{dom} c^{\bar{p}} = \bigcup_{n<\omega} \mathrm{supp}(p_n)$ such that, for each $n < \omega$ and $\alpha \in \mathrm{supp}(p_n)$, $p_n(\alpha)$ is of the form $(c_0^{\bar{p}}(\alpha), \dot{\varphi})$ where $\dot{\varphi}$ is a $\mathbb{P}'_\alpha$-name of a slalom of width $c_1^{\bar{p}}(\alpha)$.

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Technicality

For each $h : \delta \to \omega^{<\omega} \times \omega$, it can be found a single chain of ultrafilters such that it calculates limits for any uniform sequence $\bar{p}$ with $c^{\bar{p}} \subseteq h$

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Theorem (Engelking and Karłowicz 1965)

Assume CH and $\delta < (2^{\aleph_1})^+$. Then, there is a set $\{h_\varepsilon : \epsilon < \omega_1\}$ of (total) functions from δ into ω_1 such that every countable partial function from δ to ω_1 is contained in some h_ϵ.

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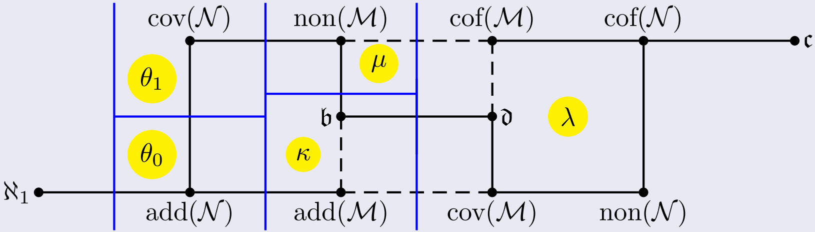
So, under CH, if $\lambda \leq 2^{\aleph_1}$ and $\mu^{\aleph_0} = \mu$ then we only need $\aleph_1$-many chains of ultrafilters.

Main Result

Theorem (Goldstern and M. and Shelah)

Let $\theta_0 \leq \theta_1 \leq \kappa = \kappa^{\aleph_0} \leq \mu = \mu^{\aleph_0}$ *be uncountable regular cardinals,* $\mu < \lambda = \lambda^{<\mu} \leq 2^\kappa$ *and assume* $\mathfrak{b} = \mathfrak{d} = \kappa$. *Then, there is a ccc poset forcing*



Question

Open question

Is it consistent that $\mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$?