Maximal trees

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Trees

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- $\exists$ largest element $\mathbf{1}$
- $\forall t \in \mathcal{T}$: $\mathrm{pred}_{\mathcal{T}}(t) = \{s \in \mathcal{T} : s \geq t\}$ is well-ordered by $\geq$ (predecessors)

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levels: $\mathrm{Lev}_\alpha(\mathcal{T}) = \{t \in \mathcal{T} : \mathrm{pred}_{\mathcal{T}}(t) \text{ has order type } \alpha\}$
height: $\mathrm{ht}(\mathcal{T}) = \min\{\alpha : \mathrm{Lev}_\alpha(\mathcal{T}) = \emptyset\}$
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$\mathcal{T} \subseteq \mathbb{P}$ is *subtree* of $(\mathbb{P}, \leq)$: $\mathbf{1} \in \mathcal{T}$ and $(\mathcal{T}, \leq \restriction (\mathcal{T} \times \mathcal{T}))$ is tree

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Note: incomparable (equivalently, incompatible) elements of $\mathcal{T}$ are not necessarily incompatible in $\mathbb{P}$.

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Fact. $\mathcal{T} \subseteq \mathbb{P}$ is maximal iff for all $p \in \mathbb{P}$

- *either* there is $q \in \mathcal{T}$ with $q \leq p$,
- *or* there are incomparable $q, r \in \mathcal{T}$ with $p \leq q, r$.

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Question (Monk)

Are there consistently maximal trees of other sizes?

Earlier Results

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Fact. $\omega_1 \leq \mathfrak{r} \leq \mathfrak{tr} \leq \mathfrak{c}$

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Let $\mathcal{F}$ be a filter on ω containing all cofinite sets

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$C \in [\omega]^\omega$ $\mathcal{F}$*-positive*: $C \cap A$ is infinite for all $A \in \mathcal{F}$

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Lemma (Extension Lemma)

Assume $\mathcal{T} = \{A_\alpha^\beta : \alpha, \beta \leq \gamma\}$ is a matrix tree. Then there is a ccc forcing end-extending $\mathcal{T}$ to a matrix tree $\mathcal{T}' = \{A_\alpha^\beta : \alpha, \beta \leq \gamma + 2\}$ such that no $C \in [\omega]^\omega$ from the ground model can be added to $\mathcal{T}'$.

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Let $A_{\gamma+2}^\beta \subseteq A_{\gamma+1}^\beta$ arbitrary.
Let $A_0^{\gamma+1} = \omega \setminus X_0$ and $A_0^{\gamma+2} = \omega \setminus X_1$.
Then $A_{\gamma+1}^\beta \cap A_0^{\beta'} = \emptyset$ for $\beta \leq \gamma$ and $\beta' \in \{\gamma + 1, \gamma + 2\}$.

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Let $A_{\gamma+2}^\beta \subseteq A_{\gamma+1}^\beta$ arbitrary.
Let $A_0^{\gamma+1} = \omega \setminus X_0$ and $A_0^{\gamma+2} = \omega \setminus X_1$.
Then $A_{\gamma+1}^\beta \cap A_0^{\beta'} = \emptyset$ for $\beta \leq \gamma$ and $\beta' \in \{\gamma+1, \gamma+2\}$.
Genericity: clause (iii) still satisfied.

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Add $A_1^{\gamma+1} \subseteq A_0^{\gamma+1}$ and $A_1^{\gamma+2} \subseteq A_0^{\gamma+2}$ by ccc forcing s.t. A_0^{β} and $A_1^{\beta'}$ are a.d. for $\beta' \in \{\gamma+1, \gamma+2\}$ and any $\beta \neq \beta'$.

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  Thus $C \subseteq^* (\omega \setminus X_0) \cap (\omega \setminus X_1) = A_0^{\gamma+1} \cap A_0^{\gamma+2}$, and C cannot be added to $\mathcal{T}'$.

Done!

Main theorem

Theorem (J. Br. 2015)

Let $\kappa \leq \lambda$ be regular uncountable cardinals with $\lambda^\omega = \lambda$. There is a ccc generic extension with $\mathfrak{tr} = \kappa$ and $\mathfrak{c} = \lambda$.

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Thus $\mathfrak{c} = \lambda$ and $\mathfrak{tr} \leq \kappa$.

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Maximality follows from Extension Lemma.

Thus $\mathfrak{c} = \lambda$ and $\mathfrak{tr} \leq \kappa$.

$\mathfrak{tr} \geq \mathfrak{r} \geq \mathrm{cov}(\mathcal{M}) \geq \kappa$ because of Cohen reals added in limit stages.

Further results and problems

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Open Problem

Let C be a set of regular cardinals (possibly satisfying some additional condition). Is there a ccc forcing extension in which $\mathrm{Spec}_{\mathrm{tree}} = C$?