Mathias–Příkrý forcing and generic ultrafilters

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Ultrafilters on ω

Definition

An ultrafilter $\mathcal{F}$ on ω is a *P-point* if for each $\mathcal{C} \in [\mathcal{F}]^\omega$ there is a pseudo-intersection $P \in \mathcal{F}$ such that $P \subset^* F$ for each $F \in \mathcal{C}$.

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An ultrafilter $\mathcal{F}$ on ω is *selective* if for each $\{A_i \colon i \in \omega\}$, a partition of ω disjoint with $\mathcal{F}$ there is a selector $S \in \mathcal{F}$ such that $|S \cap A_i| = 1$ for each $i \in \omega$.

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Let $\mathcal{F}$ be an ultrafilter in ω. The following properties are equivalent:

- $\mathcal{F}$ is selective,
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Theorem (Zapletal)

An ultrafilter $\mathcal{F}$ is a P-point iff for each analytic ideal $\mathcal{I} \subset \mathcal{F}^*$ there is an F_σ ideal $\mathcal{C}$ such that $\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{F}^*$.

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The generic filter on the poset $(\mathcal{P}(\omega) \setminus \mathsf{Fin}, \subset^*)$ *is a selective ultrafilter.*

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(LC)[1] *An ultrafilter is selective if and only if it is a generic filter on* $\mathcal{P}(\omega) \setminus \mathsf{Fin}$ *over* $L(\mathbb{R})$.

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Let $\mathcal{I}$ be an F_σ ideal on ω. Denote by $\mathbb{Q}_\mathcal{I}$ the forcing $(\mathcal{P}(\omega) \setminus \mathcal{I}, \subset^*)$.

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Theorem (Zapletal, Ch.)

(LC) $\mathcal{F}$ *is a* $\mathbb{Q}_\mathcal{I}$*-generic filter over* $L(\mathbb{R})$ *iff*

1. $\mathcal{F}$ *is a P-point disjoint with* $\mathcal{I}$, *and*
2. *for each closed set* $C \subset \mathcal{P}(\omega)$ *disjoint with* $\mathcal{F}$ *there is* $e \in \mathcal{F}^*$ *such that* $C \subseteq \langle \mathcal{I}, \{e\} \rangle$.

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Definition (Mathias forcing)

$$\mathbb{M} = \{\langle a, F\rangle\colon a \in [\omega]^{<\omega}, F \in [\omega]^{\omega}\}$$

$$\langle a, F\rangle < \langle b, H\rangle \quad \text{if} \quad b \sqsubseteq a,\ F \subset H, \text{ and } a \setminus b \subset H.$$

Definition (Mathias–Příkrý forcing)

Let $\mathcal{F}$ be a filter on ω.

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Definition (Mathias real for $\mathcal{F}$)

$g = \bigcup\{a \colon \langle a, F\rangle\} \in \mathbb{G}$, where $\mathbb{G}$ is an $\mathbb{M}(\mathcal{F})$ generic filter.

Fact

A Mathias real is a pseudo-intersection of $\mathcal{F}$ ($g \subseteq^* F$ for each $F \in \mathcal{F}$).

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$U \subset [\omega]^{<\omega}$ is an $\mathcal{F}$*-universal set* if $[F]^{<\omega} \cap U \neq \emptyset$ for each $F \in \mathcal{F}$.

Fact

$[g]^{<\omega} \cap U \neq \emptyset$ *for each $\mathcal{F}$-universal set U.*

Definition (Pseudogeneric real for $\mathcal{F}$)

Let $\mathcal{F}$ be a filter on ω. A set $m \subset \omega$ is a *pseudogeneric real for $\mathcal{F}$* if

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Theorem

If $m \subset \omega$ is a pseudogeneric real for $\mathcal{F}$ and $c \subset \omega$ is a Cohen real, then $m \cap c$ is a Mathias real for $\mathcal{F}$.

Theorem

(LC) *$\mathcal{F}$ is a $\mathbb{Q}_{\mathcal{I}}$-generic filter over $L(\mathbb{R})$ iff*

1. *$\mathcal{F}$ is a P-point disjoint with $\mathcal{I}$, and*
2. *for each closed set $C \subset \mathcal{P}(\omega)$ disjoint with $\mathcal{F}$ there is $e \in \mathcal{F}^*$ such that $C \subseteq \langle \mathcal{I}, \{e\} \rangle$.*

Proof.

Suppose $D \in L(\mathbb{R})$ is open dense in $\mathbb{Q}_{\mathcal{I}}$.
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Let $\mathcal{F}$ be a P-point. Assume there is (in some extension of V) an elementary embedding $j \colon V \to M$ such that $\mathbb{R} \cap V$ is countable in M. Then there is a Mathias real $g \in j(\mathcal{F})$ (over V).

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Pass to an extension $V[G]$ where $j\colon V \to M$ exists.
There is a Mathias real $g \in j(\mathcal{F})$.
Now $g \in D^{V[G]}$ and $D^{V[G]} \cap M = j(D)$, i.e. $g \in j(D) \cap j(\mathcal{F})$.