A model with no strongly separable MAD families

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Outline

Basic Definitions

- We say that two infinite subsets *a* and *b* of ω are *almost disjoint or a.d.* if $a \cap b$ is finite.
- We say that a family $\mathscr{A} \subset [\omega]^{\omega}$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint.
- A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.

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- We say that a family $\mathscr{A} \subset [\omega]^{\omega}$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint.
- A *Maximal Almost Disjoint family, or MAD family* is an infinite a.d. family that is not properly contained in a larger a.d. family.
- If $\mathscr{A} \subset [\omega]^{\omega}$ is an a.d. family, then $\mathcal{I}(\mathscr{A})$ denotes the ideal on ω generated by $\mathscr{A}$.

Completely Separable MAD families

Definition

An almost disjoint family $\mathscr{A}$ is said to be completely separable if for any $b \in \mathcal{I}^+(\mathscr{A})$, there is an $a \in \mathscr{A}$ with $a \subset b$

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Their existence is connected to the existence of ADRs:

Definition

Given $\mathscr{C} \subset [\omega]^{\omega}$, *we say that a family* $\mathscr{A} = \{a_c : c \in \mathscr{C}\} \subset [\omega]^{\omega}$ *is an almost disjoint refinement (ADR) of* $\mathscr{C}$ *if*

1. $\forall c \in \mathscr{C} \, [a_c \subset c]$
2. $\forall c_0, c_1 \in \mathscr{C} \left[c_0 \neq c_1 \implies \left|a_{c_0} \cap a_{c_1}\right| < \omega\right]$.

Completely Separable MAD families

Facts:

- If $\mathscr{C} \subset [\omega]^{\omega}$ has an ADR, then there is dense ideal $\mathcal{I}$ such that $\mathcal{I} \cap \mathscr{C} = 0$.
- $\mathcal{I}^{+}$ has an ADR for every dense $\mathcal{I}$ iff for every dense $\mathcal{I}$, there is a completely separable $\mathscr{A} \subset \mathcal{I}$.
- If $\mathscr{A}$ is completely separable, then for every $b \in \mathcal{I}^{+}(\mathscr{A})$, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that $a \subset b$.

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- If $\mathscr{A}$ is completely separable, then for every $b \in \mathcal{I}^{+}(\mathscr{A})$, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that $a \subset b$.

Question (Erdos-Shelah)

Is there a completely separable MAD family $\mathscr{A} \subset [\omega]^{\omega}$? Is there a completely separable MAD $\mathscr{A} \subset \mathcal{I}$ for each dense $\mathcal{I}$?

Completely Separable MAD families

- Easy to see that answer is 'yes' if $\mathfrak{a} = \mathfrak{c}$
- (Balcar, Simon, Vojtas): Yes if any one of these holds: $\mathfrak{s} = \omega_1, \mathfrak{b} = \mathfrak{d}$, or $\mathfrak{d} \leq \mathfrak{a}$
- (Balcar, Vojtas): Every non-principal ultrafilter has an ADR.

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- (Balcar, Vojtas): Every non-principal ultrafilter has an ADR.

Theorem (Shelah [2])

If $\mathfrak{c} < \aleph_\omega$, then the answer is yes.

Strongly Separable MAD families

There are several possible definitions. Let us rephrase the definition of completely separable MAD families:

Definition

Given an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$, *let us say that a set* $A \subset [\omega]^{<\omega}$ *is* $\mathcal{I}$*-positive if for every* $a \in \mathcal{I}$, $\exists s \in A \, [s \cap a = 0]$.

A set $a \in \mathcal{P}(\omega)$ is in $\mathcal{I}^+$ iff $\{\{n\} : n \in a\}$ is $\mathcal{I}$-positive.

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So $\mathscr{A}$ is completely separable iff for every $\mathcal{I}(\mathscr{A})$ positive set $A \subset [\omega]^{<\omega}$ consisting entirely of singletons, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B = a$.

Strongly Separable MAD families

Steprans-Shelah definition: for every $\mathcal{I}(\mathscr{A})$ positive set $A \subset [\omega]^{<\omega}$, there are $\mathfrak{c}$ many $a \in \mathscr{A}$ such that there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

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They applied this to the Calkin Algebra, $\mathscr{C}(\mathbb{H}) = \mathscr{B}(\mathbb{H})/\mathscr{K}(\mathbb{H})$.

Definition

A masa in a C^* algebra is a maximal, abelian, self adjoint subalgebra (C^* subalgebra).

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Definition

A masa in a C^* algebra is a maximal, abelian, self adjoint subalgebra (C^* subalgebra).

Theorem (Steprans and Shelah [3])

If there is a strongly separable MAD family (in their sense), then there is a masa in $\mathscr{C}(\mathbb{H})$ that is generated by its projections, and does not lift to a masa in $\mathscr{B}(\mathbb{H})$.

Strongly Separable MAD families

Question (Steprans and Shelah)

Is there a strongly separable MAD family (in their sense)? Is there one with the property that for every $\mathcal{I}(\mathscr{A})$ *positive set* $A \subset [\omega]^{<\omega}$*, there is at least one* $a \in \mathscr{A}$ *such that there is an infinite pairwise disjoint* $B \in [A]^{\omega}$ *so that* $\bigcup B \subset a$*? Can one be constructed if* $\mathfrak{c} < \aleph_{\omega}$*?*

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Definition

An a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ is *strongly separable* if for every $\mathcal{I}(\mathscr{A})$ positive $A \subset [\omega]^{<\omega}$ there is an $a \in \mathscr{A}$ such that there is an infinite $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

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An a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ is *strongly separable* if for every $\mathcal{I}(\mathscr{A})$ positive $A \subset [\omega]^{<\omega}$ there is an $a \in \mathscr{A}$ such that there is an infinite $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

Theorem (R.)

It is consistent that there are no strongly separable MAD families.

Metrizability of countable Fréchet groups

Definition

Recall that a topological space X is *Fréchet* if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points in A converging to p.

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Is it consistent that every countable Fréchet group is metrizable?

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Question (Malykhin)

Is it consistent that every countable Fréchet group is metrizable?

Definition

Let us say that an ideal $\mathcal{I}$ is *Fréchet* if for every $\mathcal{I}$-positive $A \subset [\omega]^{<\omega}$, there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\forall a \in \mathcal{I}\,[|a \cap (\bigcup B)| < \omega]$.

Metrizability of countable Fréchet groups

- If $\mathcal{I}$ is a Frechet ideal that is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \triangle \rangle$ by by stipulating that

$$\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega} [s \cap a = 0 \implies s \in A]\}$$

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$$\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega} [s \cap a = 0 \implies s \in A]\}$$

  is a neighborhood base at 0.
- The topology is Fréchet because a set $A \subset [\omega]^{<\omega}$ is $\mathcal{I}$ positive iff 0 is in the closure of A. And $A \subset [\omega]^{<\omega}$ has an subsequence converging to 0 iff there is an infinite pairwise disjoint $B \in [A]^{\omega}$ so that $\forall a \in \mathcal{I} [|a \cap (\bigcup B)| < \omega]$.

Metrizability of countable Fréchet groups

Question (Gruenhage and Szeptycki)

Is there an uncountable a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ such that $\mathcal{I}(\mathscr{A})$ is Fréchet?
Is there a Fréchet ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ that is not countably generated?

Theorem (Brendle and Hrusak [1])

It is consistent that no $\mathcal{I}$ with fewer than $\mathfrak{c}$ generators is Fréchet.

My proof uses a modification of the forcing of Brendle and Hrusak.

The Proof

The big picture:

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- We assume $\diamondsuit(S_1^2)$ in the ground model and we do a finite support iteration of σ-centered forcings of length $\aleph_2$.
- Given a strongly MAD family $\mathscr{A}$ in the final model, there is a club of ω_1 limits of ω_2 where the maximality of $\mathscr{A}$ reflects.

The Proof

The big picture:

- We assume $\diamond(S^2_1)$ in the ground model and we do a finite support iteration of σ-centered forcings of length $\aleph_2$.
- Given a strongly MAD family $\mathscr{A}$ in the final model, there is a club of ω_1 limits of ω_2 where the maximality of $\mathscr{A}$ reflects.
- At a stage α when $\mathscr{A}$ is maximal, we do a forcing that adds a set $A \subset [\omega]^{<\omega}$ with the following two properties:
  - At no stage $\beta \geq \alpha$ is there an almost disjoint $\mathscr{A}' \supset \mathscr{A}$ so that A is not $\mathcal{I}(\mathscr{A}')$ positive.
  - At no stage $\beta \geq \alpha$ is there an almost disjoint $\mathscr{A}' \supset \mathscr{A}$ so that there are $a \in \mathscr{A}'$ and infinite set $B \in [A]^{\omega}$ so that $\bigcup B \subset a$.

  The second requirement was met by the Brendle-Hrusak forcing.

The Proof

- Suppose $\mathscr{A}$ is a MAD family (something weaker than maximality suffices). Then $\mathcal{I}^{+}(\mathscr{A})$ is a selective coideal. Choose a selective ultrafilter $\mathcal{U} \subset \mathcal{I}^{+}(\mathscr{A})$.
- For $s \in \mathrm{FIN} = [\omega]^{\omega} \setminus \{0\}$, $\mathrm{cone}(s) = \{t \in \mathrm{FIN} : s \subset t\}$. We define

$$\mathscr{G} = \{A \subset \mathrm{FIN} : \forall b \in \mathcal{U} \exists s \in \mathrm{FIN}(b)\, [\mathrm{cone}(s) \subset A]\}.$$

- It is easy to check that $\mathscr{G}$ is a filter on FIN. The forcing is $\mathbb{P} = \mathbb{L}(\mathscr{G})$.

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- It is easy to check that $\mathscr{G}$ is a filter on FIN. The forcing is $\mathbb{P} = \mathbb{L}(\mathscr{G})$.
- $\mathbb{P}$ adds a sequence $X : \omega \to \text{FIN}$. By genericity $\text{ran}(X)$ is $\mathcal{I}(\mathscr{A})$ positive and for all $a \in \mathscr{A}$, $\forall^{\infty} n \in \omega [X(n) \not\subset a]$.

The Proof

We also need to show:

- For all $a_0, \ldots, a_k \in \mathscr{A}$, and for all $b \in \mathbf{V}[G]$ with the property that for any $n \in \omega$, if $X(n) \cap (a_0 \cup \cdots \cup a_k) = 0$, then $b \cap X(n) \neq 0$, there is $a \in \mathscr{A}$ such that $|b \cap a| = \omega$
- For all infinite $B \in [\mathrm{ran}(X)]^{\omega}$ that is in $\mathbf{V}[G]$, there is $a \in \mathscr{A}$ such that $|a \cap (\bigcup B)| = \omega$.

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Lemma

Suppose $\{\mathring{A}_n : n \in \omega\} \subset \mathbf{V}^{\mathbb{P}}$ so that for each $n \in \omega$, $\Vdash \mathring{A}_n \in \left[\mathrm{ran}(\mathring{X})\right]^\omega$. Then there is $a \in \mathscr{A}$ such that for all $n \in \omega$, $\Vdash \left|a \cap \left(\bigcup \mathring{A}_n\right)\right| = \omega$.

The Proof

Lemma

Let $\{\mathring{a}_n : n \in \omega\} \subset \mathbf{V}^{\mathbb{P}}$ *and let* $a_0, \dots, a_k \in \mathscr{A}$. *Assume that for all* $n, m \in \omega$, $\Vdash \mathring{X}(m) \cap (a_0 \cup \cdots \cup a_k) = 0 \implies \mathring{X}(m) \cap \mathring{a}_n \neq 0$. *Then there is* $a \in \mathscr{A} \setminus \{a_0, \dots, a_k\}$ *such that for all* $n \in \omega$, $\Vdash \left|a \cap \mathring{a}_n\right| = \omega$.

The Proof

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The proof uses Ramsey theory.

Lemma

Let $F : \mathrm{FIN} \to \omega$ *such that for every* $s \in \mathrm{FIN}$, $F(s) \in s$. *Then there is a set* $b \in [\omega]^\omega$ *such that either (1) or (2) holds:*

1. $\forall s \in \mathrm{FIN}\,(b) \forall c \in [b/s]^\omega \exists t \sqsubset c\, [t \neq 0 \wedge F(s \cup t) \in t]$
2. $\forall c \in [b]^\omega \exists s \sqsubset c\, [s \neq 0 \wedge \forall t \in [b/s]^{<\omega}\, [F(s \cup t \in s)]]$.

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- Put $b = \omega \setminus (a_0 \cup \cdots \cup a_k) \in \mathcal{U}$.

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- Fix $\sigma \in (\mathrm{FIN})^{<\omega}$ and $n \in \omega$. We may define a function $F_{\langle \sigma, n \rangle} : \mathrm{FIN}(b) \to b$ by $F_{\langle \sigma, n \rangle}(s)$ is the least $k \in s$ so that $\neg \exists q \in \mathbb{P} [\mathrm{stem}(q) = \sigma^\frown \langle s \rangle \wedge q \Vdash k \notin \mathring{a}_n]$

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- We can find $b_{\langle \sigma, n \rangle} \in \mathcal{U} \cap [b]^{\omega}$ which satisfies either (1) or (2)
- There is $a \in \mathscr{A} \setminus \{a_0, \ldots, a_k\}$ so that $\forall \langle \sigma, n \rangle \left[\left| a \cap b_{\langle \sigma, n \rangle} \right| = \omega \right]$. This is the a we want.

The Proof

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- We can find $b_{\langle\sigma,n\rangle} \in \mathcal{U} \cap [b]^\omega$ which satisfies either (1) or (2)
- There is $a \in \mathscr{A} \setminus \{a_0, \ldots, a_k\}$ so that $\forall\langle\sigma, n\rangle \left[\left|a \cap b_{\langle\sigma,n\rangle}\right| = \omega\right]$. This is the a we want.
- Suppose, for a contradiction, that there is $n \in \omega$, and $p \in \mathbb{P}$ and $m \in \omega$ so that $p \Vdash a \cap \mathring{a}_n \subset m$. Put $\sigma = \mathrm{stem}\,(p)$.

Questions

Conjecture

A further modification of the approach will prove the consistency of "for every uncountable a.d. family $\mathscr{A}$, $\mathcal{I}(\mathscr{A})$ is not Fréchet".

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