Lévy collapse and the precipitousness of normal ideals over ω_1

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1. Introduction

Thm. (Foreman-Magidor-Shelah)
Let μ be a supercompact cardinal. Then

$$V^{\mathrm{Col}(\omega_1,<\mu)} \models \text{“NS}_{\omega_1}\text{ is precipitous”}.$$

- $\mathrm{Col}(\omega_1, < \mu)$ is the Lévy collapse forcing μ to be ω_2.

Ishiu proved that ideals naturally defined from tail club guessing sequences are also precipitous in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

Def.
Suppose that $A \subseteq \mathrm{Lim}(\omega_1)$.
$\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle$ is called a **tail club guessing (t.c.g.)** sequence on A if

(i) c_α is unbounded subset of α of order-type ω,

(ii) For any club $C \subseteq \omega_1$ there exists $\alpha \in A$ with $c_\alpha \subseteq^* C$, i.e. $c_\alpha \setminus \beta \subseteq C$ for some $\beta < \alpha$.

Def.
For a t.c.g. sequence $\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle$, let

$\mathrm{TCG}(\vec{c})$:= the set of all $A' \subseteq \omega_1$ such that $\vec{c} \mid A'$ is not a t.c.g. sequence.
$(\vec{c} \mid A' = \langle c_\alpha \mid \alpha \in A \cap A' \rangle)$

- $\mathsf{TCG}(\vec{c})$ is a normal proper ideal over ω_1 for any t.c.g. sequence $\vec{c}$.

- $\diamondsuit$ holds in $V^{\mathsf{Col}(\omega_1,<\mu)}$. Hence there exists a t.c.g. sequence.

- If $\vec{c}$ is a t.c.g. sequence in $V^{\mathsf{Col}(\omega_1,<\mu)}$, then $\mathsf{TCG}(\vec{c})$ is not equal to a restriction of NS_{ω_1}.

Thm. (Ishiu)
Let μ be a supercompact cardinal. Then

$$V^{\mathsf{Col}(\omega_1,<\mu)} \models \text{“}\mathsf{TCG}(\vec{c}) \text{ is precipitous for any t.c.g. seq. } \vec{c}\text{”}.$$

Question

How about other ideals over ω_1 ?

What kind of ideals become precipitous in $V^{\mathrm{Col}(\omega_1,<\mu)}$?

I do not know much on this question yet.

In this talk I will present

- other examples of normal ideals which are precipitous in $V^{\mathrm{Col}(\omega_1,<\mu)}$,
- an example of normal ideal which is not precipitous in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

In the course of presenting examples of precipitous ideals,
I also introduce new reflection principles associated to ideals.

2. Examples of precipitous ideals

2.1 ideals which become equal to NS_{ω_1}

The ideals below are equal to NS_{ω_1} in $V^{\mathsf{Col}(\omega_1,<\mu)}$. Hence they are precipitous in $V^{\mathsf{Col}(\omega_1,<\mu)}$:

- the set of all $A \subseteq \omega_1$ on which a t.c.g. sequence exists
- the set of all $A \subseteq \omega_1$ on which a $\diamondsuit$-sequence exists

$\vdots$

2.2 ideals defined from guessing sequences

example 1: ideals defined from weak club guessing sequences

Def.

Suppose that $A \subseteq \mathrm{Lim}(\omega_1)$.
$\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle$ is called a **weak club guessing (w.c.g.)** sequence on A if

(i) c_α is an unbounded subset of α of order-type ω,

(ii) For any club $C \subseteq \omega_1$ there exists $\alpha \in A$ such that $c_\alpha \cap C$ unbounded in α.

Def.

For a w.c.g. sequence $\vec{c} = \langle c_\alpha \mid \alpha \in A \rangle$, let

WCG($\vec{c}$) := the set of all $A' \subseteq \omega_1$ such that $\vec{c} \mid A'$ is not a w.c.g. sequence.

- WCG($\vec{c}$) is a normal proper ideal over ω_1 for any w.c.g. sequence $\vec{c}$.

- There exists a w.c.g. sequence in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

- If $\vec{c}$ is a w.c.g. seq. in $V^{\mathrm{Col}(\omega_1,<\mu)}$, then WCG($\vec{c}$) differs from restrictions of NS_{ω_1} and TCG($\vec{c}'$) for any t.c.g. seq. $\vec{c}'$.

Thm.

Let μ be a supercompact cardinal. Then

$V^{\mathrm{Col}(\omega_1,<\mu)} \models$ "WCG($\vec{c}$) is precipitous for any w.c.g. seq. $\vec{c}$".

example 2: ideals defined from $\diamondsuit'$-sequences

For any $\diamondsuit$-sequence $\vec{b} = \langle b_\alpha \mid \alpha \in A \rangle$,

"the set of all $A' \subseteq \omega_1$ with $\vec{b} \restriction A'$ not a $\diamondsuit$-sequence"

is not an ideal.

For the following variant of $\diamondsuit^-$ the above construction gives normal ideals:

Def.
Suppose that $A \subseteq \omega_1$.
$\vec{N} = \langle N_\alpha \mid \alpha \in A \rangle$ is called a $\diamondsuit'$-sequence on A if

(i) N_α is a countable transitive model of ZFC$^-$,

(ii) for any $B \subseteq \omega_1$ the set $\{\alpha \in A \mid B \cap \alpha \in N_\alpha\}$ is stationary.

- $\diamondsuit' \Leftrightarrow \diamondsuit^- \Leftrightarrow \diamondsuit$.

Def.

For a $\diamondsuit'$-sequence $\vec{N} = \langle N_\alpha \mid \alpha \in A \rangle$, let

$$\mathrm{DMD}(\vec{N}) := \text{the set of all } A' \subseteq \omega_1 \text{ such that } \vec{N} \restriction A' \text{ is not a } \diamondsuit'\text{-sequence.}$$

- $\mathrm{DMD}(\vec{N})$ is a normal proper ideal over ω_1 for any $\diamondsuit'$-seq. $\vec{N}$.
- In $V^{\mathrm{Col}(\omega_1, <\mu)}$ there exists a $\diamondsuit'$-sequence.
- NS_{ω_1}, $\mathrm{TCG}(\vec{c})$ and $\mathrm{WCG}(\vec{c})$ are in fact presaturated in $V^{\mathrm{Col}(\omega_1, <\mu)}$. But $\mathrm{DMD}(\vec{N})$ cannot be presaturated.

Thm.

Let μ be a supercompact cardinal. Then

$$V^{\mathrm{Col}(\omega_1, <\mu)} \models \text{“}\mathrm{DMD}(\vec{N}) \text{ is precipitous for any } \diamondsuit'\text{-seq. } \vec{N}\text{”}.$$

Outline of proof of precipitousness

Roughly, the precipitousness of NS_{ω_1}, $\mathsf{TCG}(\vec{c})$, $\mathsf{WCG}(\vec{c})$ and $\mathsf{DMD}(\vec{N})$ can be proved in the same manner.
First recall that the precipitousness of NS_{ω_1} follows from the following two facts on the stationary reflection principle (SR):

Def.

SR $\equiv$ For any set $W \supseteq \omega_1$ and any stationary $X \subseteq [W]^\omega$, there exists $W' \subseteq W$ such that

(i) $|W'| = \omega_1 \subseteq W$
(ii) $X \cap [W']^\omega$ is stationary in $[W']^\omega$.

Fact (Foreman-Magidor-Shelah)
If μ is a supercompact cardinal, then SR holds in $V^{\mathsf{Col}(\omega_1,<\mu)}$.

Fact (Foreman-Magidor-Shelah)
SR implies the precipitousness of NS_{ω_1}.

Below let I be one of NS_{ω_1}, $\mathsf{TCG}(\vec{c})$, $\mathsf{WCG}(\vec{c})$ and $\mathsf{DMD}(\vec{N})$.

The precipitousness of I can be proved by generalizing the previous facts.

Step1: Define a tower of ideals $\langle I_W \mid W \supseteq \omega_1 \rangle$ associated with I so that the reflection principle with respect to this tower holds in $V^{\mathsf{Col}(\omega_1, <\mu)}$.

Step2: Prove that the reflection principle with respect to the tower implies the precipitousness of I.

- Step1 -

- Step 1 is quite the same for each $I = \mathsf{NS}_{\omega_1}, \mathsf{TCG}(\vec{c}), \mathsf{WCG}(\vec{c})$, $\mathsf{DMD}(\vec{N})$. We only use the following property of I:

  Lem.1
  Every σ-closed forcing preserves I-positive sets.

- For each σ-closed poset P let I^P denotes the ideal I defined in V^P.

  If $I = \mathsf{NS}_{\omega_1}$, then I^P is ${\mathsf{NS}_{\omega_1}}^{V^P}$.

  If $I = \mathsf{DMD}(\vec{N})$, then I^P is $\mathsf{DMD}(\vec{N})^{V^P}$.

  $\vdots$

- For each set $W \supseteq \omega_1$ define I_W as follows:

$$I_W \;:=\; \text{the set of all } X \subseteq [W]^\omega \text{ such that}$$
$$\exists P\text{: } \sigma\text{-closed poset which forces } |W| = \omega_1,$$
$$\exists \pi : \omega_1 \to W\text{: surjection in } V^P,$$
$$\{\alpha \in \omega_1 \mid \pi``\alpha \in X\} \in I^P.$$

Using Lem.1, we can easily prove the following:

Lem.2

$$I_W \;=\; \text{the set of all } X \subseteq [W]^\omega \text{ such that}$$
$$\forall P\text{: } \sigma\text{-closed poset which forces } |W| = \omega_1,$$
$$\forall \pi : \omega_1 \to W\text{: surjection in } V^P,$$
$$\{\alpha \in \omega_1 \mid \pi``\alpha \in X\} \in I^P.$$

Note

If $I = \mathrm{NS}_{\omega_1}$, then I_W is the nonstationary ideal over $[W]^\omega$.

- The lemmata below naturally follows from the definition and Lem.1 and 2. Below let W be a set $\supseteq \omega_1$:

Lem.3

I_W is a normal ideal over $[W]^\omega$.

Proof of normality

Suppose $\{X_a \mid a \in W\} \subseteq I_W$.

Let P be a σ-closed poset forcing $|W| = \omega_1$, and take a surjection $\pi : \omega_1 \to W$ in V^P. Then in V^P,

$$\begin{aligned} &\{\alpha \in \omega_1 \mid \pi``\alpha \in \nabla_{a \in W} X_a\} \\ &\quad = \nabla_{\beta < \omega_1} \{\alpha \in \omega_1 \mid \pi``\alpha \in X_{\pi(\beta)}\} \in I^P. \end{aligned}$$

Hence $\nabla_{a \in W} X_a \in I_W$. □

Lem.4

I_{ω_1} is essentially equal to I.

Lem.5

$\langle I_W \mid W \supseteq \omega_1 \rangle$ forms a tower of ideals,
i.e. if $\omega_1 \subseteq W \subseteq W'$, then for each $X \subseteq [W]^\omega$,

$$X \in I_W \quad \Leftrightarrow \quad \{x' \in [W']^\omega \mid x' \cap W \in X\} \in I_{W'}.$$

Lem.6

Every σ-closed forcing preserves I_W-positive sets,
i.e. for any σ-closed P and any $W \supseteq \omega_1$, if $X \subseteq [W]^\omega$ is I_W-positive in V, then X is I_W^P-positive in V^P.

- From Lem.6 and the usual generic ultrapower argument, it follows that the reflection principle with respect to the tower holds in $V^{\mathrm{Col}(\omega_1,<\mu)}$:

Def.

$\mathrm{RP}(I) \equiv$ For any set $W \supseteq \omega_1$ and any I_W-positive $X \subseteq [W]^\omega$, there exists $W' \subseteq W$ such that

(i) $|W'| = \omega_1 \subseteq W$
(ii) $X \cap [W']^\omega$ is $I_{W'}$-positive.

Note $\mathrm{RP}(\mathrm{NS}_{\omega_1})$ is equivalent to SR.

Prop.7

If μ is supercompact, then $\mathrm{RP}(I)$ holds in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

- **Step2** -

The following holds for each $I = \mathsf{NS}_{\omega_1}, \mathsf{TCG}(\vec{c}), \mathsf{WCG}(\vec{c})$, $\mathsf{DMD}(\vec{N})$:

Prop.8
RP(I) implies the precipitousness of I.

This can be shown by the catch-your-tail argument for each I. But the proof is slightly different from each other. I do not know a uniform proof of Prop.8.

3. Example of non-precipitous ideal

We show the following:

Prop.9

If μ is an inaccessible cardinal, then there exists a normal non-precipitous ideal over ω_1 in $V^{\mathsf{Col}(\omega_1,<\mu)}$.

- First note that

$$\mathsf{Col}(\omega_1,<\mu) \simeq \mathsf{Col}(\omega_1,<\mu) * \mathsf{Add}(\omega_1,\mu),$$

where $\mathsf{Add}(\omega_1,\mu)$ is the countable support product of $\mathsf{Add}(\omega_1)$ $(= {}^{<\omega_1}\omega_1)$ of length μ.

Moreover $\mu = \omega_2$ and CH holds in $V^{\mathsf{Col}(\omega_1,<\mu)}$.

Hence it suffices to show the following:

Prop.10

Assume CH. Then there exists a normal non-precipitous ideal over ω_1 in $V^{\mathsf{Add}(\omega_1,\omega_2)}$.

- We use canonical functions:

  For each $\eta < \omega_2$, take a surjection $\pi_\eta : \omega_1 \to \eta$, and define $h_\eta : \omega_1 \to \omega_1$ as

  $$h_\eta(\alpha) \;:=\; \text{the order type of } \pi_\eta``\alpha.$$

  We call h_η the canonical function for η.

  For any normal ideal J over ω_1, h_η represents η in the generic ultrapower by J.

Outline of Proof of Prop.10

Let G be Add(ω_1, ω_2)-generic filter. We work in $M := V[G]$.

For each $\xi < \omega_2$ let $f_\xi : \omega_1 \to \omega_1$ be the ξ-th fnt. added by G.

For each $\xi, \eta < \omega_2$ let

$$A_{\xi,\eta} \;:=\; \{\alpha \in \omega_1 \mid f_\xi(\alpha) \leq h_\eta(\alpha)\},$$

and let

$$J \;:=\; \text{the normal ideal generated by } \{A_{\xi,\eta} \mid \xi, \eta < \omega_2\}.$$

(In the generic ultrapower by J, each f_ξ represents an ordinal which is greater than all $\eta < {\omega_2}^M$.)

Then we can prove the following:

- J is a proper ideal.
- ${\omega_2}^M$ is not in the well-founded part of the generic ultrapower by J. □

4. Question

4.1 ideals defined from unbounded functions

Let h_η be the canonical function for each $\eta < \omega_2$.

- In the proof of Prop.10, each $f = f_\xi$ has the property below:

  "For any $\eta < \omega_2$ the set $\{\alpha \in \omega_1 \mid f(\alpha) > h_\eta(\alpha)\}$ is stationary."

  A (maybe partial) function $f : \omega_1 \to \omega_1$ with this property is said to be **unbounded**.

- $\diamondsuit$ implies the existence of unbounded functions.

  Def.
  For an unbounded function $f : \omega_1 \to \omega_1$, let

  $$\begin{aligned} \text{UBD}(f) &:= \text{the normal ideal generated by } \{A_\eta \mid \eta < \omega_2\} \\ &= \text{the set of all } A \subseteq \omega_1 \text{ with } f \mid A \text{ not unbounded.} \end{aligned}$$

  Here $A_\eta = \{\alpha \in \omega_1 \mid f(\alpha) \leq h_\eta(\alpha)\}$.

- I do not know whether UBD(f) is precipitous or not in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

- As is DMD($\vec{N}$), UBD(f) cannot be presaturated.

- As in the case of NS_{ω_1}, TCG($\vec{c}$), WCG($\vec{c}$) and DMD($\vec{N}$), we can construct a tower of ideals associated to UBD(f), and can prove that the reflection principle RP(UBD(f)) with respect to this tower holds in $V^{\mathrm{Col}(\omega_1,<\mu)}$.

  But I do not know whether RP(UBD(f)) implies the precipitousness of UBD(f) or not.

Question

Is UBD(f) precipitous in $V^{\mathrm{Col}(\omega_1,<\mu)}$?
Does RP(UBD(f)) imply the precipitousness of UBD(f) ?

4.2 existence of non-precipitous ideals

It is consistent that every normal ideal over ω_1 is precipitous. In fact, if NS_{ω_1} is saturated, then every normal ideal over ω_1 is a ristriction of NS_{ω_1} to some stationary set. Hence if NS_{ω_1} is saturated, then every normal ideal over ω_1 is precipitous.

Question

In what situation normal non-precipitous ideals over ω_1 exist ?

For example, $\diamondsuit$ implies the existence of normal non-precipitous ideals ?

Thank you very much.