All automorphisms of all Calkin algebras Part I$\frac{I}{II}$: The introduction

Ilijas Farah

York University

RIMS, Kyoto, November 2009

Trivial automorphisms

An automorphism Φ of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is *trivial* if there is $h\colon \mathbb{N} \to \mathbb{N}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) & \xrightarrow{X \mapsto h^{-1}(X)} & \mathcal{P}(\mathbb{N}) \\
\downarrow{\scriptstyle \pi_{\mathrm{Fin}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Fin}}} \\
\mathcal{P}(\mathbb{N})/\operatorname{Fin} & \xrightarrow[\Phi]{} & \mathcal{P}(\mathbb{N})/\operatorname{Fin}
\end{array}$$

commutes.

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *has* $2^{\mathfrak{c}}$ *automorphisms*

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *has* $2^{\mathfrak{c}}$ *automorphisms and only* $\mathfrak{c}$ *of them are trivial.*

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ *has* $2^{\mathfrak{c}}$ *automorphisms and only* $\mathfrak{c}$ *of them are trivial.*

Theorem (Shelah, 1979)

If ZFC is consistent then so is ZFC+'All automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ *are trivial.'*

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *has* $2^{\mathfrak{c}}$ *automorphisms and only* $\mathfrak{c}$ *of them are trivial.*

Theorem (Shelah–Steprāns, 1989)

PFA implies 'All automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ *are trivial.'*

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *has* $2^{\mathfrak{c}}$ *automorphisms and only* $\mathfrak{c}$ *of them are trivial.*

Theorem (Velickovic, 1989)

TA+MA implies 'All automorphisms of $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *are trivial.'*

Rigidity I

Theorem (W. Rudin, 1957)

CH implies $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has $2^{\mathfrak{c}}$ automorphisms and only $\mathfrak{c}$ of them are trivial.

Theorem (Velickovic, 1989)

TA+MA implies 'All automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial.'

TA:=Todorcevic's Axiom (aka OCA)

TA Assume X is a separable metric space and $K \subseteq [X]^2$ is open. Then either there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K$ or $X = \bigcup_{n<\omega} X_n$ so that $[X_n]^2 \cap K = \emptyset$ for all n.

Rigidity beyond $\aleph_0$

Theorem (Velickovic, 1989)

TA+MA implies all automorphisms of $\mathcal{P}(\aleph_1)/\operatorname{Fin}$ *are trivial.*

Theorem (Velickovic, 1989)

TA+MA implies all automorphisms of $\mathcal{P}(\aleph_1)/\operatorname{Fin}$ *are trivial.*
PFA implies all automorphisms of $\mathcal{P}(\kappa)/\operatorname{Fin}$ *are trivial, for every infinite cardinal* κ.

Stone duality

Boolean algebra		Topological space
$\mathcal{P}(\mathbb{N})$	$\longleftrightarrow$	$\beta\mathbb{N}$
$\mathcal{P}(\mathbb{N}/\,\mathsf{Fin})$	$\longleftrightarrow$	$\beta\mathbb{N} \setminus \mathbb{N}$

Stone duality

Boolean algebra		Topological space
$\mathcal{P}(\mathbb{N})$	$\longleftrightarrow$	$\beta\mathbb{N}$
$\mathcal{P}(\mathbb{N}/\,\mathrm{Fin})$	$\longleftrightarrow$	$\beta\mathbb{N}\setminus\mathbb{N}$

Theorem (Shelah, 1979)

If ZFC is consistent then so is ZFC+'every autohomeomorphism of $\beta\mathbb{N}\setminus\mathbb{N}$ has an extension to a continuous self-map of $\beta\mathbb{N}$.

Stone duality

Boolean algebra		Topological space
$\mathcal{P}(\mathbb{N})$	$\longleftrightarrow$	$\beta\mathbb{N}$
$\mathcal{P}(\mathbb{N}/\,\mathrm{Fin})$	$\longleftrightarrow$	$\beta\mathbb{N}\setminus\mathbb{N}$

Theorem (Shelah, 1979)

If ZFC is consistent then so is ZFC+'every autohomeomorphism of $\beta\mathbb{N}\setminus\mathbb{N}$ has an extension to a continuous self-map of $\beta\mathbb{N}$.

Theorem (Farah, 1998)

TA+MA implies that if X is a 0-dimensional locally compact Polish space then every autohomeomorphism of $\beta X\setminus X$ has an extension to a continuous self-map of βX.

More rigidity

Conjecture (Farah, 2000)

PFA implies that all isomorphisms $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *are trivial, whenever* $\mathcal{I}$ *and* $\mathcal{J}$ *are analytic ideals.*

More rigidity

Conjecture (Farah, 2000)

PFA implies that all isomorphisms $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *are trivial, whenever* $\mathcal{I}$ *and* $\mathcal{J}$ *are analytic ideals.*

Theorem (Farah, 1997–2004)

Confirmed the conjecture for many ideals, including all nonpathological ideals.

More rigidity

Conjecture (Farah, 2000)

PFA implies that all isomorphisms $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *are trivial, whenever* $\mathcal{I}$ *and* $\mathcal{J}$ *are analytic ideals.*

Theorem (Farah, 1997–2004)

Confirmed the conjecture for many ideals, including all nonpathological ideals.

Corollary

PFA implies that quotients over two nonpathological ideals are isomorphic if and only if the ideals are isomorphic.

Boolean groups

If

$$\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) & \xrightarrow{\;F\;} & \mathcal{P}(\mathbb{N}) \\
\big\downarrow{\scriptstyle \pi_{\mathrm{Fin}}} & & \big\downarrow{\scriptstyle \pi_{\mathrm{Fin}}} \\
\mathcal{P}(\mathbb{N})/\,\mathrm{Fin} & \xrightarrow[\;\Phi\;]{} & \mathcal{P}(\mathbb{N})/\,\mathrm{Fin}
\end{array}$$

commutes then we say F is a *representation* of Φ.

Boolean groups

If

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{\quad F \quad} & \mathcal{P}(\mathbb{N}) \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{Fin}}} & & \Big\downarrow{\scriptstyle \pi_{\mathrm{Fin}}} \\ \mathcal{P}(\mathbb{N})/\mathrm{Fin} & \xrightarrow[\quad \Phi \quad]{} & \mathcal{P}(\mathbb{N})/\mathrm{Fin} \end{array}$$

commutes then we say F is a *representation* of Φ.

$$(\mathcal{P}(\mathbb{N}), \Delta) \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

Boolean groups

If

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{\;F\;} & \mathcal{P}(\mathbb{N}) \\ \downarrow{\scriptstyle \pi_{\mathrm{Fin}}} & & \downarrow{\scriptstyle \pi_{\mathrm{Fin}}} \\ \mathcal{P}(\mathbb{N})/\,\mathrm{Fin} & \xrightarrow[\Phi]{} & \mathcal{P}(\mathbb{N})/\,\mathrm{Fin} \end{array}$$

commutes then we say F is a *representation* of Φ.

$$(\mathcal{P}(\mathbb{N}), \Delta) \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

A group isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is *trivial* if it has a representation that is a group homomorphism.

Theorem (Farah, 2000)

Every group isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Borel-measurable representation is trivial whenever $\mathcal{J}$ is a nonpathological ideal.

rigidity

Theorem (Farah, 2000)

Every group isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *that has a Borel-measurable representation is trivial whenever* $\mathcal{J}$ *is a nonpathological ideal.*

Lemma

If $\mathcal{I}$ *is a proper analytic ideal then* $\mathcal{P}(\mathbb{N})/\mathcal{I} \cong \mathcal{P}(\mathbb{N})$ *(as a group).*

Non-rigidity

Theorem (Farah, 2000)

Every group isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ *that has a Borel-measurable representation is trivial whenever* $\mathcal{J}$ *is a nonpathological ideal.*

Lemma

If $\mathcal{I}$ *is a proper analytic ideal then* $\mathcal{P}(\mathbb{N})/\mathcal{I} \cong \mathcal{P}(\mathbb{N})$ *(as a group).*

Proof.

Both are $\mathbb{F}_2$-vector spaces of dimension $2^{\aleph_0}$. □

Gelfand–Naimark–Segal duality

For a compact Hausdorff space X let $C(X)$ be the C*-algebra of all continuous complex-valued functions on X.

Gelfand–Naimark–Segal duality

For a compact Hausdorff space X let $C(X)$ be the C*-algebra of all continuous complex-valued functions on X.

$$\begin{array}{rcr} \beta\mathbb{N} & \longleftrightarrow & C(\beta\mathbb{N}) \\ \beta\mathbb{N}\setminus\mathbb{N} & \longleftrightarrow & C(\beta\mathbb{N}\setminus\mathbb{N}) \end{array}$$

Gelfand–Naimark–Segal duality

For a compact Hausdorff space X let $C(X)$ be the C*-algebra of all continuous complex-valued functions on X.

$$\begin{array}{rcr} \beta\mathbb{N} & \longleftrightarrow & C(\beta\mathbb{N}) \\ \beta\mathbb{N}\setminus\mathbb{N} & \longleftrightarrow & C(\beta\mathbb{N}\setminus\mathbb{N}) \end{array}$$

Theorem (Shelah, 1979)

If ZFC is consistent then so is 'all automorphisms of $C(\beta\mathbb{N}\setminus\mathbb{N})$ are 'trivial.'

The Calkin algebra

$H = \ell_2(\aleph_0)$: an infinite-dimensional complex Hibert space.
$\mathcal{B}(H)$: The algebra of bounded linear operators.
$\mathcal{K}(H)$: The ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$: Quotient C*-algebra, the so-called *Calkin algebra*.

The Calkin algebra

$H = \ell_2(\aleph_0)$: an infinite-dimensional complex Hibert space.
$\mathcal{B}(H)$: The algebra of bounded linear operators.
$\mathcal{K}(H)$: The ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$: Quotient C*-algebra, the so-called *Calkin algebra*.

Question (Brown–Douglas–Fillmore, 1977)

Are all automorphisms Φ of the Calkin algebra inner?

The Calkin algebra

$H = \ell_2(\aleph_0)$: an infinite-dimensional complex Hibert space.
$\mathcal{B}(H)$: The algebra of bounded linear operators.
$\mathcal{K}(H)$: The ideal of compact operators.
$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$: Quotient C*-algebra, the so-called *Calkin algebra*.

Question (Brown–Douglas–Fillmore, 1977)

Are all automorphisms Φ *of the Calkin algebra inner?*

As usually, Φ is *inner* if for some $u \in \mathcal{C}(H)$ we have

$$\Phi(a) = uau^*$$

for all a.

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ has $2^{\mathfrak{c}}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ has $2^{\mathfrak{c}}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ has $2^{\mathfrak{c}}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

TA implies all automorphisms of $\mathcal{C}(H)$ are inner.

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ has $2^{\mathfrak{c}}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

TA implies all automorphisms of $\mathcal{C}(H)$ are inner.

Question

What can be said in the case when H is nonseparable?

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ has $2^{\mathfrak{c}}$ automorphisms, (and only $\mathfrak{c}$ inner automorphisms).

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ then $\mathcal{C}(H)$ has $2^{\aleph_1}$ automorphisms.

Theorem (Farah, 2007)

TA implies all automorphisms of $\mathcal{C}(H)$ are inner.

Question

What can be said in the case when H is nonseparable?

Answer

All kinds of things

A rather complete picture

Theorem (Phillips–Weaver, 2006)

CH implies $\mathcal{C}(H)$ *has* $2^{\mathfrak{c}}$ *automorphisms, (and only* $\mathfrak{c}$ *inner automorphisms).*

Proposition (Farah, Geschke 2007)

If $\mathfrak{d} = \aleph_1$ *then* $\mathcal{C}(H)$ *has* $2^{\aleph_1}$ *automorphisms.*

Theorem (Farah, 2007)

TA implies all automorphisms of $\mathcal{C}(H)$ *are inner.*

Question

What can be said in the case when H *is nonseparable?*

Answer

All kinds of things but not enough.