All automorphisms of all Calkin algebras
Part I: Polish Aronzajn trees

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Aronszajn trees

Definition

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An ω_1-*tree* is a sequence X_ξ, $\xi < \omega_1$ of countable sets and commuting surjections for $\eta < \xi$

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An ω_1-tree is *special* if there exist antichains $\mathcal{A}_n$, for $n < \omega$, such that

$$\bigcup_{\xi<\omega_1} X_\xi = \bigcup_{n<\omega_1} \mathcal{A}_n.$$

Polish Aronszajn trees

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A *Polish* ω_1*-tree* is a sequence (X_ξ, d_ξ), $\xi < \omega_1$ of Polish spaces and commuting surjective contractions for $\eta < \xi$

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A Polish ω_1-tree is a *Polish Aronszajn tree* (PA-tree) if

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On a Pω_1-tree (X_ξ, d_ξ), $\xi < \omega_1$ define a pseudo-metric

$$d(x, y) = d_\eta(\pi_{\xi\eta}(x), y)$$

if $x \in X_\xi$, $y \in X_\eta$ and $\xi \geq \eta$.

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Some $\mathcal{A} \subseteq \bigcup_\xi X_\xi$ is an ε-*antichain* if $d(x, y) > \varepsilon$ for $x \neq y$ in $\mathcal{A}$.

ε-special Pω_1-trees

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A Pω_1-tree is ε-*special* if there are ε-antichains $\mathcal{A}_n$, for $n \in \mathbb{N}$, such that $(\bigcup_n \mathcal{A}_n) \cap X_\xi$ is dense in X_ξ for each $\xi < \omega_1$.

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Lemma

If a PA-tree is ε-special then it has no cofinal branches.

MA and ε-special trees

Fix a PA-tree T:

$$X_\xi, \text{ for } \xi < \omega_1 \text{ and } \pi_{\xi\eta} \colon X_\xi \to X_\eta \text{ for } \omega_1 > \xi \geq \eta.$$

Fix $\varepsilon > 0$.

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2. $d(x_\eta, x_\xi) < \varepsilon$ for all $\xi > \eta$.

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Pf. Fix a countable dense $Z_\xi \subseteq X_\xi$ for each ξ. Let

$$\mathbb{P} = \{p \in \bigcup_\xi Z_\xi : p \text{ is a finite } \varepsilon/2\text{-antichain}\}$$

ordered by $p \leq q$ iff $p \supseteq q$.

The key lemma–the right formulation

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We prove that is T has no ε-branches then $\mathbb{P}$ is powerfully ccc, i.e., $\mathbb{P}^{<\aleph_0}$ is ccc.

Proving $\mathbb{P}$ is (powerfully) ccc

Fix p_ξ, for $\xi < \omega_1$, in $\mathbb{P}$. We may assume

$$p_\xi = p \cup r_\xi$$

and that for some n

$$r_\xi = \{x_\xi^0, \ldots x_\xi^{n-1}\}.$$

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Fix a uniform ultrafilter $\mathcal{U}$ on ω_1.

Still proving $\mathbb{P}$ is (powerfully) ccc

For each ξ fix $A_\xi \in \mathcal{U}$ such that for some

$$s_\xi : n^2 \longrightarrow 2$$

all $\eta \in A_\xi$ and all $(i, j) \in n^2$ we have

$$d(x^i_\xi, x^j_\eta) < \frac{\varepsilon}{2} \text{ iff } s_\xi(i, j) = 0.$$

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Fix $A \in \mathcal{U}$ and s such that for all $\xi \in A$ we have $s_\xi = s$.

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$$d(x^i_\xi, x^i_\eta) \leq d(\xi^i, x^j_\eta) + d(x^j_\eta, x^i_\xi) < \varepsilon$$

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This tree is even special.

Subtrees and an another key lemma

A sequence Y_ξ, $\xi < \omega_1$ is a *subtree* of a Pω_1-tree $T = \langle X_\xi, \pi_{\xi\eta} \rangle$ if

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If a Pω_1-tree T is such that every subtree has an ε-branch for every $\varepsilon > 0$, then T has a branch.

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For each η the sequence x_ξ^n, for $n < \omega$, is a Cauchy sequence. Then

$$x_\xi = \lim_n x_\xi^n$$

defines a cofinal branch in T. $\square$

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. . . unless we assume something extra about T!

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An ω_1-tree T is *coherent* if $T \subseteq 2^{\omega_1}$ and for all s and t in T the set

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Then $\langle X_\xi, \pi_{\xi\eta} \rangle$ is a *Polish coherent family*.
It is *nontrivial* if

$$\varprojlim_{\xi} X_\xi = \emptyset.$$

Proposition

Assume PFA. If a Polish coherent family is nontrivial, then for some $\varepsilon > 0$ it has an ε-special ω_1-subtree.