Cardinal Invariants of Projections in the Calkin Algebra

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- $\mathfrak{v} = \min\{|A| : A \subseteq \mathscr{A} \text{ satisfying } \ldots\}$.
- $R \subseteq A \times B$, $||(A, B, R)|| = \min\{|B| : \forall a \in A \exists b \in B\ aRb\}$.

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\begin{aligned}
\mathfrak{d} &= ||(\omega^\omega, \omega^\omega, \leq^*)|| \quad (f \leq^* g \Leftrightarrow \forall^\infty n\, f(n) \leq g(n)) \\
&= ||([\omega]^\omega_{\subseteq^*(a_n)}, [\omega]^\omega_{\subseteq^*(a_n)}, \subseteq^*)|| \quad (a \subseteq^* b \Leftrightarrow |a \backslash b| < \infty) \\
\mathfrak{b} &= ||(\omega^\omega, \omega^\omega, \not\geq^*)|| \\
&= ||([\omega]^\omega_{\subseteq^*(a_n)}, [\omega]^\omega_{\subseteq^*(a_n)}, \not\supseteq^*)|| \\
\mathfrak{s} &= ||([\omega]^\omega, [\omega]^\omega, \{(a, b) : |a \cap b| = |a \cap \omega \backslash b| = \infty\})|| \\
\mathfrak{t} &= \min\{\kappa : \exists \subseteq^* \text{-decreasing tower} (t_\xi)_{\xi \in \kappa} \subseteq [\omega]^\omega\}.
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- $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{b}, \mathfrak{s} \leq \mathfrak{d} \leq \mathfrak{c}$.
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Outline

Hilbert Space Projections and Subspaces

- A *projection* P on a Hilbert space H is a linear operator s.t.

$$\forall x \in H \quad P(Px) = Px \quad \text{and} \quad Px \perp (x - Px).$$

- C is a closed subspace $\Rightarrow \exists!$ projection P_C s.t. $\mathcal{R}(P_C) = C$.
  P is a projection $\Rightarrow P = P_{\mathcal{R}(P)}$.
- P is a projection $\Leftrightarrow P \in \mathcal{P}(\mathcal{B}(H)) \Leftrightarrow P^2 = P$ and $P = P^*$
  (For $T \in \mathcal{B}(H)$ $\langle Tx, y\rangle = \langle x, T^* y\rangle$)
- For closed subspaces U and V of H,

$$U \subseteq V \Leftrightarrow P_U \leq P_V \Leftrightarrow P_U P_V = P_U = P_V P_U.$$

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Modulo Finite Dimension/Rank

- Define $U \subseteq^\bullet V \Leftrightarrow \mathrm{codim}_U(U \cap V) < \infty$.

$$\begin{aligned} U \subseteq^\bullet V &\Leftrightarrow \mathrm{rank}(P_U - P_U P_V) < \infty \\ &\Leftrightarrow \pi_0(P_U P_V) = \pi_0(P_U) = \pi_0(P_V P_U) \\ &\Leftrightarrow \pi_0(P_U) \leq \pi_0(P_V). \end{aligned}$$

$$\mathcal{K}_0(H) = \{T \in \mathcal{B}(H) : \dim(\mathcal{R}(T)) < \infty\}$$

$$\pi_0 : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}_0(H)$$

- Problem: $\mathcal{K}_0(H)$ not closed $\Rightarrow$ no norm on $\mathcal{B}(H)/\mathcal{K}_0(H)$.

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Modulo Compact Operators

- $\mathcal{K}(H) = \{T \in \mathcal{B}(H) : \overline{T[B_1(H)]} \text{ is compact}\} = \overline{\mathcal{K}_0(H)}$.
  Calkin (C*-)algebra of H is $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$.
  $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ is the canonical homomorphism.
- $p \in \mathcal{P}(\mathcal{C}(H)) \Leftrightarrow \exists P \in \mathcal{P}(\mathcal{B}(H))\ \pi(P) = p$.
- Define $U \subseteq^* V \Leftrightarrow \pi(P_U) \leq \pi(P_V) \Leftrightarrow P_U - P_U P_V$ compact.

$$U \subseteq^* V \Leftrightarrow \forall \epsilon > 0 \exists \text{ fin codim } W \subseteq U\ s.t. ||P_V^{\perp} P_W|| < \epsilon.$$

- Fix a basis $(e_\lambda)_{\lambda \in \Lambda}$ of H. For $A \subseteq \Lambda$, $V_A = \overline{\text{span}}\{e_\lambda : \lambda \in A\}$.

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1. Projections
2. **Order Properties**
3. Cardinal Invariants

To G.L.B. Or Not To G.L.B. (I)

- $\mathcal{P}(\mathcal{C}(H))$ is not a lattice (assuming $\dim(H) \not< \infty$).
- Example: Take disjoint $(A_n) \subseteq [\omega]^\omega$ with $\bigcup A_n = \omega$ and set

$$\begin{aligned} U &= \overline{\mathrm{span}}\{e_{2n} : n \in \omega\} = V_{\{2n : n \in \omega\}} \\ W_k &= \overline{\mathrm{span}}\{e_{2n} + (1/k)e_{2n+1} : n \in A_k\}. \end{aligned}$$

U and $W = \bigoplus W_k$ have no g.l.b.: Given $f \in \omega^\omega$ set

$$\begin{aligned} F_k^f &= \{2n : n \in A_k \wedge n < f(k)\} \quad \text{and} \\ V^f &= V_{\bigcup F_k^f} \subseteq V_{\{2n : n \in \omega\}} = U. \end{aligned}$$

Then $\mathrm{codim}_{V^f}(V_{\bigcup_{k \geq n} F_k}) = \sum_{k<n} \dim(F_k) < \infty$ and $||P_W^\perp P_{V_{\bigcup_{k \geq n} F_k}}|| = 1/(1+n^2) \to 0 \therefore V_f \subseteq^* W \ (\forall f \in \omega^\omega)$

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Actually, $\forall V \subseteq^* U, W\ \exists f \in \omega^\omega\ V \subseteq^* V^f$.

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As $(\omega^\omega, \leq^*)$ has no max, U and W have no g.l.b.. □

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  Step 2: (V_n) $\subset^*$-decreasing $\Rightarrow$ collective g.l.b. $\cong (\omega^\omega, \leq^*)$.
- Corollary: $\mathfrak{b}^* = \mathfrak{b}$ and $\mathfrak{d}^* = \mathfrak{d}$.

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Essential Spectrum

$$
\begin{array}{rcl}
\sigma_e(S) & = & \sigma(\pi(S)) \\
& = & \{r \in \mathbb{R} : \forall \delta > 0 \ \mathrm{rank}(E(r+\delta) - E(r-\delta)) \not< \infty\} \\
& = & \{r \in \mathbb{R} : \mathrm{rank}(E(r) - E(r-)) \not< \infty \vee r \in \overline{\sigma(S)\setminus\{r\}}\} \\
(s,t] & \subseteq & \mathbb{R}\setminus\sigma_e(S) \quad \Leftrightarrow \quad \mathrm{rank}(E_S(t) - E_S(s)) < \infty \\
\therefore \ s & = & \sup(\sigma_e(PQP) \cap [0,1)) < 1 \\
& \Rightarrow & \forall r \in [s,1) E_{PQP}(r) =^* E_{PQP}(s) \\
& \Rightarrow & \mathcal{R}(P) \wedge^* \mathcal{R}(Q) = \mathcal{R}(E_{PQP}(s))^\perp \\
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& \Rightarrow & \mathcal{R}(P) \text{ and } \mathcal{R}(Q) \text{ have no g.l.b.}
\end{array}
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- Define $P^\perp = 1 - P = P_{\mathcal{R}(P)^\perp}$. Note $P^\perp P = 0$.
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- Say $PQPv = rv$, $r \in (0,1)$. Note $v = r^{-1}PQPv \in \mathcal{R}(P)$.

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- $\mathcal{P}(\mathcal{C}(H))$ is separative.

$$\begin{aligned} P \wedge^* Q \neq 0 \quad &\Leftrightarrow \quad \forall r < 1 \ \mathrm{rank}(E_{PQP}(r)^\perp) \not< \infty \\ &\Leftrightarrow \quad \|\pi(PQ)\|^2 = \|\pi(PQP)\| \\ &\quad\quad = \max(\sigma_e(PQP)) = 1. \end{aligned}$$

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Splitting

- Each $\mathscr{P}(\omega)/\mathrm{Fin}$ cardinal invariant has $(\geq)$2 analogs $\because$

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  $\mathfrak{s}^\perp = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(\mathcal{B}(l^2))$ is a strongly splitting family$\}$.

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Splitting

- Each $\mathscr{P}(\omega)/\mathrm{Fin}$ cardinal invariant has ($\geq$)2 analogs $\because$

$$\begin{array}{ccc} P \wedge^* Q^\perp \neq 0 & \Rightarrow & P \not\leq^* Q \\ \Updownarrow & \not\Leftarrow & \Updownarrow \\ ||\pi(PQ^\perp)|| = 1 & & ||\pi(PQ^\perp)|| > 0 \end{array}$$

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Block Subspaces

- V is a block subspace of $H = \mathrm{span}(e_n)$ means $\exists$ IP (I_n) and $\exists (v_n) \subseteq l^2$ s.t. $V = \mathrm{span}(v_n)$ and $\forall n\ v_n \in \mathrm{span}\{e_k : k \in I_n\}$.
- Block subspaces are $\subseteq^*$-dense.
  Given inf dim $V \subseteq H$ recursively pick unit vectors $(v_n) \subseteq V$

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\begin{aligned}
v_0 &= (0, \frac{1}{5}, \frac{3}{4}, \frac{1}{2}, \frac{1}{10}, \ldots) \quad \text{(arbitrary)} \\
v_1 &= (0,0,0,0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{16}}, \ldots) \in V \cap l^{2\perp}_{k_0}, \ k_0 >> 0 \\
v_2 &= (0,0,0,0 \ldots, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots) \in V \cap l^{2\perp}_{k_1}, \ k_1 >> k_0 \\
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$V \supseteq \mathrm{span}(v_n) =^* $ block subspace. $\square$

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Interval Paritions

- Consequence: CIs on $\mathcal{P}(\mathcal{C}(H))$ often related to IP CIs.
- Eg. $A \subseteq \omega$ splits IP (I_n) $\Leftrightarrow$ $\exists^\infty n\ I_n \subseteq A$ and $\exists^\infty n\ I_n \subseteq \omega \setminus A$.

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  $\Rightarrow \mathfrak{s}^\perp \leq \mathfrak{s}^{\mathrm{IP}}$.
- $\mathfrak{s}^{\mathrm{IP}} = \max(\mathfrak{s}, \mathfrak{b})$ (A. Kamburelis, B. Weglorz (1995)).

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  Consistently no towers in $\mathcal{I}^* \; \forall$ such ideals $\mathcal{I}$ (Brendle).

  Corollary: Consistently all towers in $[\omega]^\omega$ remain towers.
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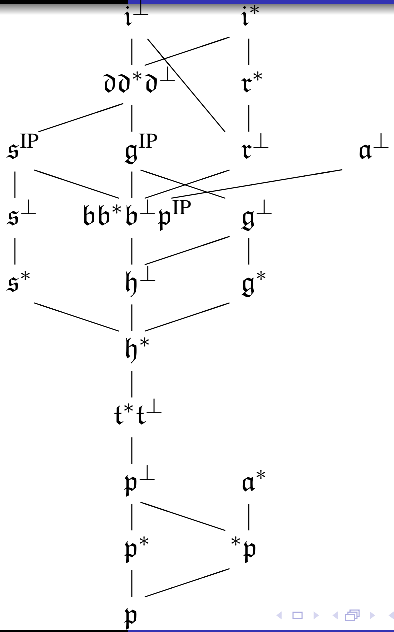
$\mathfrak{i}^{\perp}$
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$\mathfrak{d}\mathfrak{d}^{*}\mathfrak{d}^{\perp}$
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