Forcing consequences of PFA together with the continuum large

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PFA implies $2^{\aleph_0} = \aleph_2$.

All known proofs of this implication use forcing notions that collapse ω_2.

Question: Does *FA*$(\{\mathbb{P} : \mathbb{P} \text{ proper and cardinal–preserving}\})$ imply $2^{\aleph_0} = \aleph_2$? Does even *FA*$(\{\mathbb{P} : \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\})$ imply $2^{\aleph_0} = \aleph_2$?

In the first part of the talk I will isolate a certain subclass Γ of $\{\mathbb{P} : \mathbb{P} \text{ proper}, |\mathbb{P}| = \aleph_1\}$ and will sketch a proof that $\mathit{FA}(\Gamma) + 2^{\aleph_0} > \aleph_2$ is consistent.

$\mathit{FA}(\Gamma)$ will be strong enough to imply for example the negation of Justin Moore's $\mho$ and other strong forms of the negation of Club Guessing.

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$FA(\Gamma)$ will be strong enough to imply for example the negation of Justin Moore's $\mho$ and other strong forms of the negation of Club Guessing.

Notation

If N is a set such that $N \cap \omega_1 \in \omega_1$, set $\delta_N = N \cap \omega_1$.

Let X be a set. If $\mathcal{W} \subseteq [X]^{\aleph_0}$ and N is a set, $\mathcal{W}$ is an *N–unbounded subset of* $[X]^{\aleph_0}$ if for every $x \in N \cap X$ there is some $M \in \mathcal{W} \cap N$ with $x \in M$.

If $\mathbb{P}$ is a partial order, $\mathbb{P}$ is *nice* if

(a) conditions in $\mathbb{P}$ are functions with domain included in ω_1, and

(b) if $p, q \in \mathbb{P}$ are compatible, then the greatest lower bound r of p and q exists, $dom(r) = dom(p) \cup dom(q)$, and $r(\nu) = p(\nu) \cup q(\nu)$ for all $\nu \in dom(r)$ (where $f(\nu) = \emptyset$ if $\nu \notin dom(f)$).

Exercise: Every set–forcing for which $glb(p, q)$ exists whenever p and q are compatible conditions is isomorphic to a nice forcing.

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More notation

Given a nice partial order $(\mathbb{P}, \leq)$, a $\mathbb{P}$–condition p and a set M such that δ_M exists, we say that *M is good for p* iff $p \restriction \delta_M \in \mathbb{P}$ and, letting

$$X = \{s \in \mathbb{P} \cap M \,:\, s \leq p \restriction \delta_M,\ s \text{ compatible with } p\},$$

(i) $X \neq \emptyset$, and

(ii) for every $s \in X$ there is some $t \leq s$, $t \in M$, such that for all $t' \leq t$, if $t' \in M$, then $t' \in X$.

A class of posets

Let $\mathbb{P}$ be a nice poset and κ an infinite cardinal. $\mathbb{P}$ is κ–suitable if there are a binary relation R and a club $C \subseteq \omega_1$ with the following properties.

(1) If $p\,R\,(N, \mathcal{W})$, then the following conditions hold.

(1.1) N is a countable subset of $H(\kappa)$, $\mathcal{W}$ is an N–unbounded subset of $[H(\kappa)]^{\aleph_0}$, and all members of $\mathcal{W} \cap N$ are good for p.

(1.2) If p' is a $\mathcal{P}$–condition extending p, then there is some $\mathcal{W}' \subseteq \mathcal{W}$ such that $p'\,R\,(N, \mathcal{W}')$.

(1.3) If $\mathcal{W}' \subseteq \mathcal{W}$ is N–unbounded, then $p\,R\,(N, \mathcal{W}')$.

(1.4) $p \restriction \delta_N \in N$, and for all N' and all $\mathcal{W}'$ with $\delta_{N'} < \delta_N$,

$$p\,R\,(N', \mathcal{W}') \text{ if and only if } p \restriction \delta_N\,R\,(N', \mathcal{W}')$$

A class of posets

(2) For every $p \in \mathcal{P}$ and every finite set $\{(N_i, \mathcal{W}_i) : i < m\}$ such that

- (∘) each N_i is a countable subset of $H(\kappa)$ containing p, $\omega_1^{N_i} = \omega_1$, $\delta_{N_i} \in C$, $N_i \models ZFC^*$, and
- (∘) each $\mathcal{W}_i$ is N_i–unbounded

there is a condition $q \in \mathcal{P}$ extending p and there are $\mathcal{W}_i' \subseteq \mathcal{W}_i$ ($i < m$) such that $q \, R \, (N_i, \mathcal{W}_i')$ for all $i < m$.

We will say that a nice partial order is *absolutely κ–suitable* if it is κ–suitable in every ground model W containing it and such that $\omega_1^W = \omega_1$.

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A class of posets

Let Γ_κ denote the class of all absolutely κ–suitable posets consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$.

Easy: For all $\kappa \geq \omega_2$, $\Gamma_\kappa \subseteq$ *Proper*.

FA(Γ_κ): For every $\mathbb{P} \in \Gamma_\kappa$ and every collection $\mathcal{D}$ of size $\aleph_1$ consisting of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

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One application of $FA(\Gamma_\kappa)$: Ω

Definition (Moore) $\mho$: There is a sequence $\langle g_\delta : \delta < \omega_1 \rangle$ such that each $g_\delta : \delta \longrightarrow \omega$ is continuous with respect to the order topology and such that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ with g_δ "$C = \omega$.

(○) Club Guessing implies $\mho$.

(○) $\mho$ preserved by ccc forcing, and in fact by ω–proper forcing.

(○) Each of *BPFA* and *MRP* implies $\Omega := \neg\mho$.

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- ($\circ$) Club Guessing implies $\mho$.
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Theorem (Moore) $\mho$ implies the existence of an Aronszajn line which does not contain any Contryman suborder.

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Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies Ω.

Proof sketch:

Notation: Given X, a set of ordinals, and δ, an ordinal, set

- ($\circ$) $rank(X, \delta) = 0$ iff δ is not a limit point of X, and
- ($\circ$) $rank(X, \delta) > \eta$
  if and only if δ is a limit of ordinals ϵ such that $rank(X, \epsilon) \geq \eta$.

Given a sequence $\mathcal{G} = \langle g_\delta \,:\, \delta < \omega_1 \rangle$ of continuous colourings, let $\mathbb{P}_\mathcal{G}$ be the following poset:

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Conditions in $\mathbb{P}_{\mathcal{G}}$ are pairs $p = (f, \langle k_\xi : \xi \in D \rangle)$ satisfying the following properties:

(1) f is a finite function that can be extended to a normal function $F : \omega_1 \longrightarrow \omega_1$.

(2) For every $\xi \in dom(f)$, $rank(f(\xi), f(\xi)) \geq \xi$.

(3) $D \subseteq dom(f)$ and for every $\xi \in D$,

(3.1) $k_\xi < \omega$,
(3.2) $g_{f(\xi)}$“$range(f) \subseteq \omega \setminus \{k_\xi\}$, and
(3.3) $rank(\{\gamma < f(\xi) : g_{f(\xi)}(\gamma) \neq k_\xi\}, f(\xi)) = rank(f(\xi), f(\xi))$.

Given conditions $p_\epsilon = (f_\epsilon, (k^\epsilon_\xi : \xi \in D_\epsilon)) \in \mathbb{P}_\mathcal{G}$ for $\epsilon \in \{0, 1\}$, p_1 extends p_0 iff

(i) $f_0 \subseteq f_1$,

(ii) $D_0 \subseteq D_1$, and

(iii) $k^1_\xi = k^0_\xi$ for all $\xi \in D_0$.

Easy: If G is $\mathbb{P}_\mathcal{G}$–generic and $C = range(\bigcup\{f : (\exists \vec{k})(\langle f, \vec{k}\rangle \in G)\})$, then C is a club of ω_1^V and for every $\delta \in C$ there is $k_\delta \in \omega$ such that g_δ“$C \subseteq \omega \setminus \{k_\delta\}$.

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$\mathbb{P}_{\mathcal{G}} \in \Gamma_\kappa$ for every $\kappa \geq \omega_2$:

($\bullet$) We may easily translate $\mathbb{P}_{\mathcal{G}}$ into a nice forcing consisting of finite functions contained in $\omega_1 \times [\omega_1]^{<\omega}$.

($\bullet$) Given $p = (f, \langle k_\xi : \xi \in D\rangle) \in \mathbb{P}_{\mathcal{G}}$, $N \subseteq H(\kappa)$ countable such that $N \models ZFC^*$ and δ_N exists, and given $\mathcal{W}$ an N–unbounded set, set

$$p\, R\, (N, \mathcal{W})$$

if and only if

(a) δ_N is a fixed point of f,

(b) $\delta_N \in D$, and

(c) for every $M \in \mathcal{W}$, $g_{\delta_N}(\delta_M) \neq k_{\delta_N}$.

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Easy to verify:

(1) in the definition of κ–suitable

Let us check (2) in the definition of κ–suitable (with $C = \omega_1$)

[that is:

(2) For every $p \in \mathcal{P}$ and every finite set $\{(N_i, \mathcal{W}_i) : i < m\}$ such that

(a) each N_i is a countable subset of $H(\kappa)$ containing p, $\omega_1^{N_i} = \omega_1$, $\delta_{N_i} \in C$, $N_i \models ZFC^*$, and

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Let $(\delta_j)_{j<n}$ be the increasing enumeration of $\{\delta_{N_i} : i < m\}$.

Suppose $\{N_i : \delta_{N_i} = \delta_0\} = \{N_0, N_1, N_2\}$.
Let $\{k_0, \ldots k_3\}$ be $3 + 1 = 4$ colours not touched by g_{δ_0}“$range(f)$.

There is $k^0 \in \{k_0, \ldots k_3\}$ such that, for all $i < 3$, $\mathcal{W}'_i = \{M \in \mathcal{W}_i : \delta_M \neq k^0\}$ is N_i–unbounded.

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Now we continue with δ_1,
and get a colour k^1 we may avoid in the colouring g_{δ_1}. And so on.

In the end there is a condition $q = (f', \langle k'_\xi : \xi \in D' \rangle)$, $q \leq p$, and N_i–unbounded $\mathcal{W}'_i \subseteq \mathcal{W}_i$ ($i < m$) such that

(a) f' has all δ_j ($j < n$) as fixed points and makes the promise k^j at each δ_j, and

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Hence, $\mathbb{P}_{\mathcal{G}}$ is (isomorphic to) a forcing in Γ_κ.

An application of $FA(\{\mathbb{P}_{\mathcal{G}}\})$ gives now a witness of Ω for $\mathcal{G}$.
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Given $n < \omega$, $\mho_n$ is the following weakening of $\mho$:

$\mho_n$: There is a sequence $\langle g_\delta : \delta < \Omega_1 \rangle$ with $g_\delta : \delta \longrightarrow n$ continuous and such that for every club $C \subseteq \omega_1$ there is some $\delta \in C$ such that $g_\delta^{-1}(i) \cap C \subseteq \delta$ unbounded for each $i < n$.

$$\mho \rightarrow \ldots \rightarrow \mho_4 \rightarrow \mho_3 \rightarrow \mho_2$$

Question: Does any $\mathrm{FA}(\Gamma_\kappa)$ imply $\neg\mho_n$ for any $n < \omega$?

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Other applications of $FA(\Gamma_\kappa)$

Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies:

$\neg VWCG$: For every $\mathcal{C}$, if

(a) $|\mathcal{C}| = \aleph_1$ and
(b) for all $X \in \mathcal{C}$, $X \subseteq \omega_1$ and $ot(X) = \omega$,

then there is a club $C \subseteq \omega_1$ such that $|X \cap C| < \omega$ for all $X \in \mathcal{C}$.

$\neg VWCG$ is equivalent to the following statement:

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Proposition: For every $\kappa \geq \omega_2$, $FA(\Gamma_\kappa)$ implies Miyamoto's $Code(\text{even–odd})$.

$Code(\text{even–odd})$: For every ladder system $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ and every $B \subseteq \omega_1$ there are clubs $C, D \subseteq \omega_1$ such that for every $\delta \in D$,

(a) if $\delta \in B$, then $|C \cap A_\delta|$ is an even integer, and

(b) if $\delta \notin B$, then $|C \cap A_\delta|$ is an odd integer.

Note: $Code(\text{even–odd})$ implies $\neg WCG$.

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Note: $Code(\text{even–odd})$ implies $\neg WCG$.

The main theorem

Theorem 1 (*CH*) Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. Then there is a partial order $\mathcal{P}$ such that

(1) $\mathcal{P}$ is proper,

(2) $\mathcal{P}$ has the $\aleph_2$–chain condition,

(3) $\mathcal{P}$ forces

- ($\bullet$) $FA(\Gamma_\kappa)_{<cf(\kappa)}$
- ($\bullet$) $2^{\aleph_0} = \kappa$

We don't know of interesting consequences of $FA(\Gamma_\kappa)_{<cf(\kappa)}$ which do not already follow from $FA(\Gamma_\kappa)$ (except for $2^{\aleph_0} \geq cf(\kappa)$).

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Proof sketch

Let $\Phi : \kappa \longrightarrow H(\kappa)$ be a bijection.

(Φ exists by $2^{<\kappa} = \kappa$.)

Also, let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be this increasing sequence of regular cardinals: $\theta_0 = (2^\kappa)^+$, $\theta_\gamma = (sup_{\alpha<\gamma}\theta_\alpha)^+$ if γ is a nonzero limit ordinal, and $\theta_{\alpha+1} = (2^{\theta_\alpha})^+$.

Proof sketch (continued)

Coherent systems of structures

$\{N_i : i < m\}$ is a *coherent systems of structures* if

$a1)$ $m < \omega$ and every N_i is a countable subset of $H(\kappa)$ such that $(N_i, \in, \Phi \cap N_i) \preccurlyeq (H(\kappa), \in, \Phi)$.

$a2)$ Given distinct i, i' in m, if $\delta_{N_i} = \delta_{N_{i'}}$, then there is an isomorphism

$$\Psi_{N_i,N_{i'}} : (N_i, \in, \Phi \cap N_i) \longrightarrow (N_{i'}, \in, \Phi \cap N_{i'})$$

Furthermore, $\Psi_{N_i,N_{i'}}$ is the identity on $\kappa \cap N_i \cap N_{i'}$.

Proof sketch (continued)

$a3)$ For all i, j in m, if $\delta_{N_j} < \delta_{N_i}$, then there is some $i' < m$ such that $\delta_{N_{i'}} = \delta_{N_i}$ and $N_j \in N_{i'}$.

$a4)$ For all i, i', j in m, if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $N_{j'} = \Psi_{N_i,N_{i'}}(N_j)$.

Proof sketch (continued)

Our forcing will be the direct limit $\mathcal{P}_\kappa$ of a sequence $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$ of posets such that

- ($\circ$) $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ if $\alpha < \beta \leq \kappa$, and
- ($\circ$) a condition q in $\mathcal{P}_\alpha$ is an α–sequence p together with a certain system Δ_q of side conditions.

Unlike in a usual iteration, p will not consist of names, but of well–determined objects (finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$).

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Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$

$\mathcal{P}_0$: Conditions are $p = \{(N_i, 0) : i < m\}$ where $\{N_i : i < m\}$ is a coherent system of structures.

$\leq_0$ is $\supseteq$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

Suppose $\mathcal{P}_\alpha$ defined and suppose conditions in $\mathcal{P}_\alpha$ are pairs (p, Δ_p) with p an α–sequence and $\Delta_p = \{(N, \beta_i) : i < m\}$.

Suppose $\mathcal{P}_\alpha$ has the $\aleph_2$–chain condition and $|\mathcal{P}_\alpha| = \kappa$.

By $\kappa^{\aleph_1} = \kappa$ we may fix an enumeration $\dot{\mathcal{Q}}_i^\alpha$ (for $i < \kappa$) of nice κ–suitable partial orders consisting of finite functions included in $\omega_1 \times [\omega_1]^{<\omega}$ such that for every $\mathcal{P}_\alpha$–name $\dot{\mathcal{Q}}$ for such a poset there are κ–many $i < \kappa$ such that $\Vdash_{\mathcal{P}_\alpha} \dot{\mathcal{Q}} = \dot{\mathcal{Q}}_i^\alpha$.

We also fix $\mathcal{P}_\alpha$–names $\dot{R}_i^\alpha$ and $\dot{C}_i^\alpha$ (for $i < \kappa$) such that $\mathcal{P}_\alpha$ forces that $\dot{R}_i^\alpha$ and $\dot{C}_i^\alpha$ witness that $\dot{\mathcal{Q}}_i^\alpha$ is κ–suitable.

Let $\mathcal{M}_\alpha$ be the club of all countable elementary substructures of $H(\theta_\alpha)$ containing $\langle \mathcal{P}_\beta : \beta \leq \alpha \rangle$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

Suppose $\mathcal{P}_\alpha$ defined and suppose conditions in $\mathcal{P}_\alpha$ are pairs (p, Δ_p) with p an α–sequence and $\Delta_p = \{(N, \beta_i) : i < m\}$.

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Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

$\mathcal{P}_{\alpha+1}$: Conditions are

$$q = (p^\frown \langle f_i : i \in a \rangle, \{(N_i, \beta_i) : i < m\})$$

satisfying the following conditions. (We denote $\{(N_i, \beta_i) : i < m\}$ by Δ_q)

*b*1) For all $i < m$, $\beta_i \leq min\{\alpha + 1, sup(N_i \cap \kappa)\}$.

*b*2) The restriction of q to α is a condition in $\mathcal{P}_\alpha$. This restriction is defined as $q|_\alpha := (p, \{(N_i, \beta_i^\alpha) : i < m\})$; where $\beta_i^\alpha = \beta_i$ if $\beta_i < \alpha + 1$, and $\beta_i^\alpha = \alpha$ if $\beta_i = \alpha + 1$.

*b*3) a is a finite subset of κ.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

*b*4) For each $i \in a$, f_i is a finite function included in $\omega_1 \times [\omega_1]^{<\omega}$ and $q|_\alpha$ forces (in $\mathcal{P}_\alpha$) that $f_i \in \dot{\mathcal{Q}}_i^\alpha$.

*b*5) For every N such that $(N, \alpha + 1) \in \Delta_q$ and $\alpha + 1 \in N$, $q|_\alpha$ forces that there is some $\mathcal{W}_N \subseteq \mathcal{W}^\alpha$ such that

$$f_i \dot{R}_i^\alpha (N, \mathcal{W}_N)$$

for all $i \in a \cap N$.

Here, $\mathcal{W}^\alpha$ denotes the collection of all M such that $(M, \alpha) \in \Delta_u$ for some $u \in \dot{G}_\alpha$ and such that $M = M^* \cap H(\kappa)$ for some $M^* \in \mathcal{M}_\alpha$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

Given conditions

$$q_\epsilon = (p_\epsilon ^\frown \langle f_i^\epsilon : i \in a_\epsilon \rangle, \{(N_i^\epsilon, \beta_i^\epsilon) : i < m_\epsilon\})$$

(for $\epsilon \in \{0, 1\}$), we will say that $q_1 \leq_{\alpha+1} q_0$ if and only if the following holds.

$c1)$ $q_1|_\alpha \leq_\alpha q_0|_\alpha$

$c2)$ $a_0 \subseteq a_1$

$c3)$ For all $i \in a_0$, $q|_\alpha$ forces in $\mathcal{P}_\alpha$ that $f_i^1 \leq_{\dot{\mathcal{Q}}_i^\alpha} f_i^0$.

$c4)$ For all $i < m_0$ there exists $\widetilde{\beta}_i \geq \beta_i^0$ such that $(N_i^0, \widetilde{\beta}_i) \in \Delta_{q_1}$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

Suppose $\alpha \leq \kappa$ is a nonzero limit ordinal.

$\mathcal{P}_\alpha$ Conditions are $q = (p, \{(N_i, \beta_i) : i < m\})$ such that:

d 1) p is a sequence of length α.

d 2) For all $i < m$, $\beta_i \leq min\{\alpha, sup(X_i \cap \kappa)\}$. (Note that β_i is always less than κ, even when $\alpha = \kappa$.)

d 3) For every $\varepsilon < \alpha$, the restriction $q|_\varepsilon := (p \restriction \varepsilon, \{(X_i, \beta_i^\varepsilon) : i < m\})$ is a condition in $\mathcal{P}_\varepsilon$; where $\beta_i^\varepsilon = \beta_i$ if $\beta_i \leq \varepsilon$, and $\beta_i^\varepsilon = \varepsilon$ if $\beta_i > \varepsilon$.

d 4) The set of $\zeta < \alpha$ such that $p(\zeta) \neq \emptyset$ is finite.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

Given conditions $q_1 = (p_1, \Delta_1)$ and $q_0 = (p_0, \Delta_0)$ in $\mathcal{P}_\alpha$, $q_1 \leq_\alpha q_0$ if and only if:

e) For every $\beta < \gamma$, $q_1|_\beta \leq_\beta q_0|_\beta$. (Notice that $(p_1, \Delta_1) \leq_\gamma (p_0, \Delta_0)$ implies that for every $(X_i, \beta_i) \in \Delta_0$ there exists $\widetilde{\beta}_i \geq \beta_i$ such that $(X_i, \widetilde{\beta}_i) \in \Delta_1$.)

Notation: If $\alpha \leq \kappa$ and $q = (p, \{(N_i, \beta_i) : i < m\}) \in \mathcal{P}_\alpha$, we set $\mathcal{X}_q = \{N_i : i < m\}$.

Defining $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (continued)

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Main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$

Lemma Let $\alpha \leq \beta \leq \kappa$.

If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|_\alpha$, then $(p^\frown(r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_s)$ is a condition in $\mathcal{P}_\beta$ extending s.

Therefore, $\mathcal{P}_\alpha$ can be seen as a complete suborder of $\mathcal{P}_\beta$.

Lemma For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

Main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$

Lemma Let $\alpha \leq \beta \leq \kappa$.

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Lemma For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

Lemma Suppose $\alpha \leq \kappa$ and $N^* \in \mathcal{M}_\alpha$. Then,

$(1)_\alpha$ for every $q \in N^* \cap \mathcal{P}_\alpha$ there is $q' \leq_\alpha q$ such that $(N^* \cap H(\kappa), \alpha) \in \Delta_{q'}$, and

$(2)_\alpha$ for every $q \in \mathcal{P}_\alpha$, if $(N^* \cap H(\kappa), \alpha) \in \Delta_q$, then q is $(N^*, \mathcal{P}_\alpha)$–generic.

The proof is by induction on α.

Proof sketch of $(2)_\alpha$ in the case $\alpha = \sigma + 1$:

Let $N = N^* \cap H(\kappa)$. Let A be a maximal antichain of $\mathcal{P}_\alpha$ in N^*. By the $\aleph_2$–condition of $\mathcal{P}_\alpha$ and $cf(\kappa) \geq \omega_2$, $A \in N$.

It suffices to show that every q satisfying the hypothesis of $(2)_\alpha$ is compatible with some condition in $A \cap N^*$ $(= A \cap N)$.

By pre–density of A we may assume, without loss of generality, that q extends some condition $\tilde{q}$ in A.

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Claim

For every $i \in \kappa \backslash N$ *there are ordinals* $\alpha_i < \beta_i$ *such that*

(a) $\alpha_i \in N$ *and* $\beta_i \in (\kappa \cap N) \cup \{\kappa\}$,

(b) $\alpha_i < i < \beta_i$, *and*

(c) $[\alpha_i, \beta_i) \cap N' \cap N = \emptyset$ *whenever* $N' \in \mathcal{X}_q \backslash N^*$ *is such that* $\delta_{N'} < \delta_N$.

[This is proved using the fact that all $\Psi_{\overline{N},N}$ fix $\kappa \cap \overline{N} \cap N$ and are continuous (for $\overline{N} \in \mathcal{X}_q$ with $\delta_{\overline{N}} = \delta_N$), meaning that $\Psi_{\overline{N},N}(\xi) = sup(\Psi_{\overline{N},N}``\xi)$ whenever $\xi \in \overline{N}$ is an ordinal of countable cofinality.]

Suppose $a^q \setminus N^* = \{i_0, \dots i_{n-1}\}$, and for each $k < n$ let $\alpha_k < \beta_k$ be ordinals realizing the above claim for i_k.

Let us work in $V^{\mathcal{P}_\sigma \restriction (q|_\sigma)}$. By condition $b5)$ in the definition of $\mathcal{P}_{\sigma+1}$ we know that there is a an N–unbounded $\mathcal{W}_N \subseteq \mathcal{W}^\sigma$ such that $f_i^q \, \dot{R}_i^\sigma \, (N, \mathcal{W}_N)$ for all $i \in a^q \cap N$.

By an inductive construction (using (1) in the definition of κ–suitable) we may find an N–unbounded $\mathcal{W} \subseteq \mathcal{W}_N$ such that $f_i^q \, \dot{R}_i^\sigma \, (N, \mathcal{W})$ for all $i \in a^q \cap N$ and such that each $M \in \mathcal{W}$ is good for f_j^q for every $j \in a^q \cap M$.

Hence, we may find $M \in N$ such that

(a) $M = M^* \cap H(\kappa)$ for some $\mathcal{M}_\sigma$,

(b) M contains A, $\{N' : \alpha \in N', (N', \alpha) \in D_q \cap N\}$, $a^q \cap N^*$, $f_i^q \upharpoonright \delta_N$ for every $i \in a^q \cap N$, α_k for every $k < n$, and β_k for every $k < n$ with $\beta_k < \kappa$,

(c) $(M, \sigma) \in \Delta_u$ for some $u \in \dot{G}_\sigma$, and

(d) M is good for f_i^q for every $i \in a^q \cap N$.

For every $i \in a^q \cap N$ let f_i be a $\dot{\mathcal{Q}}_i^\sigma$–condition in M extending $f_i^q \restriction \delta_M = f_i^q \restriction \delta_N$ and such that every $\dot{\mathcal{Q}}_i^\sigma$–condition in M extending f_i is compatible with f_i^q.

By extending q below σ we may assume that $(M, \sigma) \in \Delta_q$ and that q_σ decides f_i for every $i \in a^q$.

The result of replacing f_i^q with $glb(f_i, f_i^q)$ in q for every $i \in a^q \cap N^*$ is a $\mathcal{P}_{\sigma+1}$–condition.

Hence, by further extending q if necessary we may assume that every $\dot{\mathcal{Q}}_i^\sigma$–condition in M^* extending $f_i^q \restriction \delta_M$ is compatible with f_i^q.

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Let now G be a $\mathcal{P}_\sigma$–generic filter over the ground model with $q|_\sigma \in G$.
By correctness of $M^*[G]$ within $H(\theta_\sigma)[G]$ we know that in $M^*[G]$ there is a condition q° satisfying the following conditions.

(i) $q^\circ \in A$ and $q^\circ|_\sigma \in G$.

(ii) $a^{q^\circ} = (a^{\tilde{q}} \cap N) \cup \{i_0^\circ, \ldots i_{n-1}^\circ\}$ with $\alpha_k < i_k^\circ < \beta_k$ for all $k < n$.

(iii) For all $i \in a^{\tilde{q}} \cap N^*$, $f_i^{q^\circ}$ extends $f_i^q \restriction \delta_N$ in $\dot{\mathcal{Q}}_i^\sigma$.

(iv) For every N' with $\alpha \in N'$, if $(N', \alpha) \in \Delta_q \cap N$ or $(N', \alpha) \in \Delta_{q^\circ}$, then there is an N'–unbounded $\mathcal{W}_{N'} \subseteq \mathcal{W}_\sigma$ such that

- ($\circ$) $f_i^q \restriction \delta_N \, \dot{R}_i^\sigma \, (N', \mathcal{W}_{N'})$ for all $i \in (a^q \backslash a^{\tilde{q}}) \cap M$ with $f_i^q \restriction \delta_N \notin N'$, and
- ($\circ$) $f_i^{q^\circ} \, \dot{R}_i^\sigma \, (N', \mathcal{W}_{N'})$ for all $i \in a^{q^\circ} \cap N'$.

(The existence of such a q° is witnessed, in $V[G]$, by q itself. It is expressed by saying "there is some $q^\circ \in \dot{A}$" for a suitable $\mathcal{P}_\sigma$–name $\dot{A} \in M$ definable from A, $\Delta_q \cap N$ and $f_i^q \upharpoonright \delta_M$, for $i \in a^q \cap N$).

By induction hypothesis, $q|_\sigma$ is $(M^*, \mathcal{P}_\sigma)$–generic. Hence, $M^*[G] \cap V = M^*$. It follows that q° is in M^*.

By extending q below σ we may assume that q decides q° and also that it extends $q^\circ|_\sigma$. The proof in this case will be finished if we show that q and q° are compatible.

It is not difficult to find f_i^* (for $i \in a^q \cup \{i_0^\circ, \dots i_{n_1}^*\}$) extending f_i^q and/or $f_{i_k^\circ}^{q^\circ}$ (for $k < n$) for which, in $V^{\mathcal{P}_\sigma \restriction (q|_\sigma)}$, we can verify condition $b5$) with respect to all N' such that $(N', \alpha) \in \Delta_q \cup \Delta_{q^\circ}$ and $\alpha \in N'$.

If $\delta_{N'} \geq \delta_N$, we use condition (2) (and (1)) in the definition of κ–suitable.

If $\delta_{N'} < \delta_N$ and $N' \in M^*$ (that is, $(N', \sigma + 1) \in \Delta_{q^\circ}$), we use condition (1) in the definition of κ–suitable.

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The only potentially problematic case is when $\delta_{N'} < \delta_N$ and $N' \in \mathcal{X}_q \backslash M^*$. But we are safe also in this case since then $(a^q \cup \{i_0^\circ, \ldots i_{n_1}^*\}) \cap N' = a^q \cap N'$. We apply again (1) in the definition of κ–suitable.

Finally we extend q below σ once more to a condition q' deciding f_i^*. Now we amalgamate q' and q° and get a legal $\mathcal{P}_\alpha$–condition (note that in extending q below σ we are not adding new pairs $(N', \sigma + 1)$ to Δ).

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This finishes the (very sketchy) proof of the lemma in this case. $\square$

Given ordinals $\alpha < \kappa$ and $i < \kappa$, we let $\dot{G}_i^\alpha$ be a $\mathcal{P}_{\alpha+1}$ for the collection of all f_i^q, where $q \in \dot{G}_{\alpha+1}$, $\alpha \in Psupp(q)$, and $i \in a^q$.

Lemma

For every $\alpha < \kappa$ and every $i < \kappa$, $\mathcal{P}_{\alpha+1}$ forces that $\dot{G}_i^\alpha$ is a $V^{\mathcal{P}_\alpha}$–generic filter over $\dot{\mathcal{Q}}_i^\alpha$.

From the above lemmas it is easy to see by standard arguments that $\mathcal{P}_\kappa$ forces $FA(\Gamma_\kappa)_{<cf(\kappa)}$ and $2^{\aleph_0} = \kappa$. $\square$

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Separating consequences of $FA(\Gamma_\kappa)$ (in conjunction with $2^{\aleph_0} = \aleph_2$)

Strong Club Guessing (SCG): There is a stationary set $S \subseteq \omega_1$ and a ladder system $\langle A_\delta : \delta \in S \rangle$ on S such that for every club $C \subseteq \omega_1$ there exists a club $D \subseteq C$ with the property that for every δ in $S \cap D$, a final segment of A_δ is included in C.

Note: If there is an *SCG*–sequence on S, then there is a *strong* $\mho$*–sequence on* S: a sequence of continuous functions $g_\delta : \delta \longrightarrow \omega$ ($\delta \in S$) such that for every club $C \subseteq \omega_1$, there exists a club $D \subseteq C$ with the property that for every $\delta \in D \cap S$ and every $n \in \omega$, there are cofinally many $\varepsilon \in C \cap \delta$ with $g_\delta(\varepsilon) = n$.

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Fact: There is a proper poset forcing *CH* together with the existence of an *SCG*(*Lim*(ω_1))–sequence.

Theorem 2 (*CH* + strong $\mho$) Let κ be a cardinal such that $\kappa^{\aleph_1} = \kappa$. Then there is a poset $\mathcal{P}$ such that

(1) $\mathcal{P}$ is proper and has the $\aleph_2$–chain condition, and

(2) $\mathcal{P}$ forces *Code*(even–odd), $\mho$, and $2^{\aleph_0} = \kappa$.

Fact: There is a proper poset forcing *CH* together with the existence of an *SCG*(*Lim*(ω_1))–sequence.

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Proof sketch: Let $\langle g_\delta : \delta \in S\rangle$ be a strong $\mho$–sequence. Define a "streamlined version" of the construction for Theorem 1, considering only the natural posets with finite conditions for forcing instances of $Code(\text{even–odd})$. Argue that $\langle g_\delta : \delta \in S\rangle$ remains a $\mho$–sequence in the end.

Another separation

A ladder system $\mathcal{A} = \langle A_\delta : \delta \in S \rangle$ is a *strong* WCG–*sequence* in case for every club $C \subseteq \omega_1$ there is a club $D \subseteq C$ with the property that $|A_\delta \cap C| < \aleph_0$ for every $\delta \in D \cap S$.

Theorem 3 (*CH*) Let κ be a cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$. Suppose $\mathcal{A} = \langle A_\delta : \delta \in S \rangle$ is a strong WCG–sequence with S stationary. Then there exists a proper forcing notion with the $\aleph_2$–chain condition and forcing the following statements.

(1) $\mathcal{A}$ is a WCG–sequence.

(2) $\neg\mho$

(3) $2^{\aleph_0} = \kappa$

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Ishiu has separated *WCG* from $\mho$ in both directions (and more).
In his models $2^{\aleph_0} \leq \aleph_2$.

Another strong failure of Club Guessing

Definition (Moore): *Measuring*: For every sequence $(C_\delta : \delta < \omega_1)$ such that each C_δ is a closed subset of δ there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of D,

(a) either a tail of $D \cap \delta$ is contained in C_δ,

(b) or a tail of $D \cap \delta$ is disjoint from C_δ.

- *Measuring* follows from *BPFA* and also from *MRP*.

- *Measuring* implies the negation of Weak Club Guessing and implies $\neg\mho_2$ (and hence also $\neg\mho$).

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A strong form of Measuring

Definition: Given a cardinal λ, *Measuring*$^*_{<\lambda}$ is the following statement:

For every set $\mathcal{C}$ consisting of closed subsets of ω_1 and with $|\mathcal{C}| < \lambda$ there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of D and every $C \in \mathcal{C}$,

(a) either a tail of $D \cap \delta$ is contained in C,

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Measuring$^*_{<\omega_2}$ clearly implies Measuring and $\neg$VWCG.

Measuring$^*_{<\omega_2}$ follows from *BPFA*. Measuring$^*_{<\omega_3}$ doesn't (note that Measuring$^*_{<\lambda}$ implies $2^{\aleph_0} \geq \lambda$).

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Given a cardinal $\mu \geq \omega_1$, say that a forcing notion $\mathbb{P}$ is ${}^{\mu}$*proper* if for every regular $\theta > |trcl(\mathbb{P})|$, every elementary substructure N of $H(\theta)$ of size μ containing $\mathbb{P}$ and every $p \in \mathbb{P} \cap N$, if ${}^{\omega}N \subseteq N$, then there is an $(N, \mathbb{P})$–generic condition $q \in \mathbb{P}$ extending p.

Note: If $\mu^{\aleph_0} = \mu$ and $\mathbb{P}$ is a ${}^{\mu}$proper poset, then forcing with $\mathbb{P}$ preserves all stationary sets consisting of ordinals of cofinality μ.

We do not know how to derive Measuring from any "natural" forcing axiom that we can force together with the continuum large.

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Theorem 4 Let $\lambda \leq \kappa$ be uncountable cardinals such that λ is regular, $\mu^{\aleph_0} = \mu$ for all uncountable regular cardinal $\mu < \lambda$, $2^{<\kappa} = \kappa$, and $\kappa^{<\lambda} = \kappa$. Then there exists a forcing notion $\mathcal{P}$ with the following properties.

(1) $\mathcal{P}$ is proper and ${}^{\mu}$proper for every uncountable regular cardinal $\mu < \lambda$

(2) $\mathcal{P}$ has the λ–chain condition. (From (1) and (2), together with the assumption that $\mu^{\aleph_0} = \mu$ for every uncountable regular $\mu < \lambda$, it follows that $\mathcal{P}$ preserves all cofinalities.)

(3) $\mathcal{P}$ forces Measuring$^{*}_{<\lambda}$.

(4) $\mathcal{P}$ forces $2^{\aleph_0} = \kappa$.