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ON A PROBLEM OF SPECKER ABOUT EUCLIDEAN REPRESENTATIONS OF FINITE GRAPHS

L. NGUYEN VAN THÉ

Abstract. Say that a graph G is *representable in* $\mathbb{R}^n$ if there is a map f from its vertex set into the Euclidean space $\mathbb{R}^n$ such that $\|f(x)-f(x')\| = \|f(y)-f(y')\|$ iff $\{x,x'\}$ and $\{y,y'\}$ are both edges or both non-edges in G. We prove that if G finite is neither complete nor independent, it is representable in $\mathbb{R}^{|G|-2}$. A similar result is also derived in the case of finite complete edge-colored graphs.

1. Introduction

Given a (simple and loopless) graph G and a natural number $n \in \mathbb{N}$, say that G is *representable in* $\mathbb{R}^n$ if there is a map f from the vertex set of G (which we will also denote by G in the sequel) into the Euclidean space $\mathbb{R}^n$ such that $\|f(x)-f(x')\| = \|f(y)-f(y')\|$ iff $\{x,x'\}$ and $\{y,y'\}$ are both edges or both non-edges in G. Classical results about 2-distance sets in Euclidean spaces [B81] show that if G is representable in $\mathbb{R}^n$, then $|G| \leq \binom{n+2}{2}$ where $|G|$ denotes of vertices of G. Equivalently:

$$\sqrt{|G|-\frac{3}{2}}-\frac{3}{4} \leq n \ .$$

On the other hand, it has been known for a long time[1] that every finite graph is representable in $\mathbb{R}^{|G|-1}$. It is also clear that if G is complete or independent, then G is not representable in $\mathbb{R}^{|G|-2}$ and dimension $|G|-1$ is necessary. But what about the converse? If G is neither complete nor independent, is it representable in $\mathbb{R}^{|G|-2}$? According to Pouzet, who mentions it in [P79] in connection to the famous Ulam reconstruction problem, this question was asked by Specker before 1972. The purpose of this note is to prove[2]:

Theorem 1. *Let G be a finite graph. Assume that G is neither complete nor independent. Then G is representable in $\mathbb{R}^{|G|-2}$.*

More generally, given a complete edge-colored graph (G,λ) (a complete graph G together with a map $\lambda : G^2 \longrightarrow \mathbb{R}$ such that $\lambda(x,x)=0$ and $\lambda(y,x)=\lambda(x,y)$) and

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[1] To our knowledge, this result appeared first in [R84] together with several other results about Euclidean representations of graphs. It is also a consequence of Schoenberg's theorem quoted below.

[2] We have to admit that due to the lack of references we were able to find about the question, it could very well be that the result is not new. We feel however that even in that case, the present note may serve as a useful reference about it in the future.

$n \in \mathbb{N}$, say that G is *representable in* $\mathbb{R}^n$ when there is a map $f : G \longrightarrow \mathbb{R}^n$ such that

$$\|f(x) - f(x')\| = \|f(y) - f(y')\| \text{ iff } \lambda(x, x') = \lambda(y, y') \ .$$

Again, known results about k-distance sets in Euclidean spaces [BBS83] show that if (G, λ) is representable in $\mathbb{R}^n$, then $|G| \leq \binom{k+n}{k}$. On the other hand, every finite (G, λ) is representable in $\mathbb{R}^{|G|-1}$ and if λ takes only one value, then (G, λ) is representable in $\mathbb{R}^{|G|-1}$ but not $\mathbb{R}^{|G|-2}$. But if λ takes at least two values, representability in $\mathbb{R}^{|G|-2}$ is always guaranteed:

Theorem 2. *Let (G, λ) be a complete colored graph. Assume that λ takes at least two different values. Then G is representable in $\mathbb{R}^{|G|-2}$.*

Note that Theorem 1 is a simple consequence of Theorem 2 when λ takes at most two values. Theorem 2 is proved using the following well-known result due to Schoenberg providing a criterion for representability of a complete edge-labelled graph in Euclidean spaces: for a matrix $M = (m_{ij})_{1 \leq i,j \leq n}$, define

$$Q_M = \max \left\{ \sum_{1 \leq i < j \leq n} m_{ij} x_i x_j : \sum_{k=1}^{n} x_k^2 = 1 \text{ and } \sum_{k=1}^{n} x_k = 0 \right\} \ .$$

Theorem (Schoenberg [S38])**.** *Let (G, d) be a complete edge-colored graph where $G = \{v_k : 1 \leq k \leq |G|\}$ and d takes positive values. Let $M = (d(v_i, v_j)^2)_{1 \leq i,j \leq |G|}$. Then (G, d) is isometric to a subset of $\mathbb{R}^{|G|-1}$ iff $Q_M \leq 0$. In that case, the dimension of the affine space spanned by (G, d) is $(|G| - 1)$ iff $Q_M < 0$.*

The paper is organized as follows: for the sake of clarity, we start in section 2 with a proof of Theorem 1. The scheme of the proof is then reproduced in section 3 to prove Theorem 2.

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2. Proof of Theorem 1

Let G be a finite graph that is neither complete nor independent. Enumerate the vertices of $G = \{v_k : 1 \leq k \leq |G|\}$ and let $M_G = (m_{ij})_{1 \leq i,j \leq |G|}$ denote the adjacency matrix of G with respect to this enumeration, ie:

$$m_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{M}$ be the adjacency matrix of the complement of G (the graph obtained from G by changing all the edges between different vertices into non-edges and vice-versa). For $\alpha, \beta > 0$, let

$$M(\alpha, \beta) = \alpha M + \beta \overline{M} \ .$$

Denoting $M(\alpha, \beta) = (m_{ij}^{\alpha\beta})_{1 \leq i,j \leq |G|}$, say that $M(\alpha, \beta)$ *codes a representation of G in* $\mathbb{R}^{|G|-1}$ when the complete edge-colored graph (G, d), with $d(v_i, v_j) = m_{ij}^{\alpha\beta}$, is isometric to a subset of $\mathbb{R}^{|G|-1}$. According to Schoenberg's theorem, we need to show that there are $\alpha \neq \beta > 0$ such that $Q_{M(\alpha^2, \beta^2)} = 0$.

Claim 1. *There are* $\alpha_0, \beta_0 > 0$ *such that* $Q_{M(\alpha_0^2, \beta_0^2)} > 0$.

Proof. Assume towards a contradiction that $Q_{M(\alpha^2,\beta^2)} \leq 0$ for all $\alpha, \beta > 0$. We show that G is complete or independent. Indeed, first take $\alpha, \beta > 0$ such that $2\alpha < \beta$. Since $Q_{M(\alpha^2,\beta^2)} \leq 0$, Schoenberg's theorem guarantees that $M(\alpha, \beta)$ codes a representation of G in $\mathbb{R}^{|G|-1}$ and by triangle inequality, no triangle with two sides of length α and one side of length β appears in this representation. Therefore, G does not contain the graph H drawn in Figure 1.

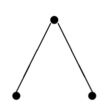

FIGURE 1. The graph H.

Similarly, choosing $2\beta < \alpha$, no triangle with one side of length α and two sides of length β appears in the representation coded by $M(\alpha, \beta)$. Therefore, G does not contain the graph K depicted in Figure 2.

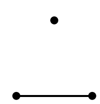

FIGURE 2. The graph K.

It follows that G is complete or independent, a contradiction. $\square$

Claim 2. *The map* $M \mapsto Q_M$ *is continuous (*$n \times n$ *matrices are seen as elements of* $\mathbb{R}^{n^2}$ *equipped with the standard topology).*

Proof. Since the topology of $\mathbb{R}^{n^2}$ is the topology induced by the ℓ_1 norm (ie $\|M\| = \sum_{1 \leq i,j \leq n} |m_{ij}|$), it is enough to show that $|Q_M - Q_N| \leq \|M - N\|$. This is done by observing that whenever $\sum_{k=1}^n x_k^2 = 1$, we have

$$\begin{aligned} \left| \sum_{1 \leq i < j \leq n} m_{ij} x_i x_j - \sum_{1 \leq i < j \leq n} n_{ij} x_i x_j \right| &\leq \sum_{1 \leq i < j \leq n} |m_{ij} - n_{ij}| |x_i x_j| \\ &\leq \sum_{1 \leq i < j \leq n} |m_{ij} - n_{ij}| \\ &\leq \|M - N\| \ . \end{aligned}$$

Therefore

$$
\begin{aligned}
Q_M &= \max\left\{ \sum_{1\le i<j\le n} m_{ij}x_ix_j : \sum_{k=1}^n x_k^2 = 1 \ \text{ and } \ \sum_{k=1}^n x_k = 0 \right\} \\
&\le \max\left\{ \sum_{1\le i<j\le n} n_{ij}x_ix_j + \|M-N\| : \sum_{k=1}^n x_k^2 = 1 \ \text{ and } \ \sum_{k=1}^n x_k = 0 \right\} \\
&\le Q_N + \|M-N\| \ .
\end{aligned}
$$

Hence, $Q_M - Q_N \le \|M-N\|$ and by symmetry, $Q_N - Q_M \le \|M-N\|$. It follows that $|Q_M - Q_N| \le \|M-N\|$. $\square$

By Claim 1 pick $\alpha_0, \beta_0 > 0$ such that $Q_{M(\alpha_0^2,\beta_0^2)} > 0$. Note that without loss of generality, we may assume that $\alpha_0 \neq \beta_0$. This is because continuity of the map $M \mapsto Q_M$ proved in Claim 2 implies continuity of $(\alpha,\beta) \mapsto Q_{M(\alpha,\beta)}$. For $t \in [0,1]$, consider the matrix

$$M\left(1+t(\alpha_0^2-1), 1+t(\beta_0^2-1)\right) \ .$$

It defines a continuous curve from $M(1,1)$ to $M(\alpha_0^2,\beta_0^2)$, and the map

$$\psi : t \mapsto Q_{M\left(1+t(\alpha_0^2-1),1+t(\beta_0^2-1)\right)}$$

is continuous on $[0,1]$. Observe that $M(1,1)$ codes the equilateral metric space on $|G|$ points where all the distances are equal to one. This metric space is Euclidean and spans an affine space of dimention $|G|-1$, therefore $\psi(0) = Q_{M(1,1)} < 0$. Observe on the other hand that $\psi(1) = Q_{M(\alpha_0^2,\beta_0^2)} > 0$. So by the intermediate value theorem, there is $\tau \in (0,1)$ such that $\psi(\tau) = 0$. That means

$$Q_{M\left(1+\tau(\alpha_0^2-1),1+\tau(\beta_0^2-1)\right)} = 0 \ .$$

So set $\alpha = \sqrt{1+\tau(\alpha_0^2-1)}$ and $\beta = \sqrt{1+\tau(\beta_0^2-1)}$. Then $\alpha \neq \beta > 0$ and $M(\alpha,\beta)$ codes a representation of G in $\mathbb{R}^{|G|-2}$. $\square$

3. Proof of Theorem 2

The proof follows exactly the same pattern as the proof of Theorem 1 so we only emphasize the ideas. Let (G,λ) be a complete colored graph where λ has range $\{l_1,\ldots,l_p\}$ of size at least two. Enumerate the vertices of $G = \{v_k : 1 \le k \le |G|\}$ and let M_i denote the adjacency matrix of the graph obtained from G by keeping only the edges with color l_i. For $\alpha_1,\ldots,\alpha_p > 0$, let

$$M(\alpha_1,\ldots,\alpha_p) = \sum_{i=1}^p \alpha_i M_i \ .$$

According to Schoenberg's theorem, we need to show that there are distinct $\alpha_1,\ldots,\alpha_p > 0$ such that $Q_{M(\alpha_1^2,\ldots,\alpha_p^2)} = 0$.

Claim. *There are $a_1,\ldots,a_p > 0$ such that $Q_{M(a_1^2,\ldots,a_p^2)} > 0$.*

Proof. Suppose not. Then $Q_{M(\alpha_1^2,\ldots,\alpha_p^2)} \le 0$ for all $\alpha_1,\ldots,\alpha_p > 0$. Varying the coefficients $\alpha_1,\ldots,\alpha_p$ and taking, turn by turn, α_i much larger than all the other coefficients, triangle inequality in the corresponding representations shows that all the triangles in (G,λ) must have all their egdes of the same color. Therefore, λ only takes one value, a contradiction. $\square$

So pick $a_1, \ldots, a_p > 0$ such that $Q_{M(a_1^2,\ldots,a_p^2)} > 0$. Note that the continuity of the map $M \mapsto Q_M$ (Claim 2) guarantees that without loss of generality, we may assume that all the a_i's are distinct. For $t \in [0,1]$, consider the matrix

$$M\left(1 + t(a_1^2 - 1), \ldots, 1 + t(a_p^2 - 1)\right) \ .$$

It defines a continuous curve from $M(1, \ldots, 1)$ to $M(a_1^2, \ldots, a_p^2)$, and the map

$$\psi : t \mapsto Q_{M\left(1+t(a_1^2-1),\ldots,1+t(a_p^2-1)\right)}$$

is continuous on $[0,1]$. Since $M(1,\ldots,1)$ codes a Euclidean metric space that spans an affine space of dimention $|G|-1$, we have $\psi(0) = Q_{M(1,\ldots,1)} < 0$. On the other hand, $\psi(1) = Q_{M(a_1^2,\ldots,a_p^2)} > 0$. So by the intermediate value theorem, there is $\tau \in (0,1)$ such that $\psi(\tau) = 0$. That means

$$Q_{M\left(1+\tau(a_1^2-1),\ldots,1+\tau(a_p^2-1)\right)} = 0 \ .$$

So for $1 \leq i \leq p$, set $\alpha_i = \sqrt{1 + \tau(a_i^2 - 1)}$. Then all the α_i's are > 0 and distinct, and $M(\alpha_1, \ldots, \alpha_p)$ codes a representation of G in $\mathbb{R}^{|G|-2}$. □

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Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada, T2N1N4.

E-mail address: `nguyen@math.ucalgary.ca`